# ON RECURRENCE FORMULAE FOR SUMS INVOLVING BINOMIAL COEFFICIENTS

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## 1. A SURVEY ON RECURRENCE FORMULAE FOR SUMS WITH BINOMIAL COEFFICIENTS AND A NEW FIVE-TERM FORMULA

Since *Apery* [1] has proved the irrationality of $\zeta(3)$ in 1979 by a linear series transformation with powers of binomial coefficients, many mathematicians are interested in linear recurrences for finite sums of powers of binomial coefficients. Apery's proof is based on the recurrence formula

$$(n+1)^3\,u_{n+1} \;-\; (34n^3+51n^2+27n+5)\,u_n \;+\; n^3\,u_{n-1} \;=\; 0 \tag{1}$$

for the sum

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 .$$

*R. Askey* and *J.A.Wilson* [2] found a three-term recurrence formula for

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+a+d}{k+d} \binom{n+k+b+e}{k+e} \binom{n+k+c+f}{k+f},$$

where $a+d=b+c$. From this identity one can derive (1) by taking

$$a=b=c=d=e=f=0\,.$$

It is well-known that ...

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^2 \quad \text{satisfies a four-term recurrence formula:}$$

$$(59n+35)(n-1)^2u_{n-2} + (295n^3-120n^2-60n+35)u_{n-1}$$
$$+(2301n^3+1365n^2-376n-240)u_n - 2(59n-24)(n+1)^2u_{n+1} \;=\; 0\ ;$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \quad \text{satisfies a three-term recurrence formula, [3]:}$$

$$n^2u_{n-1} + (11n^2+11n+3)u_n - (n+1)^2u_{n+1} \;=\; 0\ ;$$

$$\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k} \quad \text{satisfies a five-term recurrence formula, [2]:}$$

$$p_1(n)u_{n-2} + p_2(n)u_{n-1} + p_3(n)u_n + p_4(n)u_{n+1} + p_5(n)u_{n+2} \;=\; 0$$

with integer polynomials $p_1, p_2, p_3, p_4, p_5$ of degree 8;

$\sum_{k=0}^{n} \binom{n}{k}^3$ satisfies a three-term recurrence formula, [2], [7]:

$$8n^2 u_{n-1} + (7n^2 + 7n + 2)u_n - (n+1)^2 u_{n+1} \;=\; 0\ ;$$

$\sum_{k=0}^{n} \binom{n}{k}^4$ satisfies a three-term recurrence formula, [2], [8]:

$$4n(4n-1)(4n+1)u_{n-1} + 2(2n+1)(3n^2+3n+1)u_n - (n+1)^3 u_{n+1} \;=\; 0\ .$$

*M.A. Perlstadt* [13] has proved four-term recurrences for

$$\sum_{k=0}^{n} \binom{n}{k}^5 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k}^6$$

by using a computer, namely

$$\begin{aligned}
&32(55n^2 + 33n + 6)(n-1)^4 u_{n-2} \\
&-(19415n^6 - 27181n^5 + 7453n^4 + 3289n^3 - 956n^2 - 276n + 96)u_{n-1} \\
&-(1155n^6 + 693n^5 - 732n^4 - 715n^3 + 45n^2 + 210n + 56)u_n \\
&+(55n^2 - 77n + 28)(n+1)^4 u_{n+1} \;=\; 0
\end{aligned}$$

and

$$\begin{aligned}
&24(6n-7)(2n-1)(6n-5)(91n^3 + 91n^2 + 35n + 5)(n-1)^3 u_{n-2} \\
&-(153881n^9 - 307762n^8 + 185311n^7 + 2960n^6 - 31631n^5 - 88n^4 + 5239n^3 - 610n^2 \\
&\qquad -440n + 100)u_{n-1} \\
&-n(3458n^8 + 1729n^7 - 2947n^6 - 2295n^5 + 901n^4 + 1190n^3 + 52n^2 - 228n - 60)u_n \\
&+n(91n^3 - 182n^2 + 126n - 30)(n+1)^5 u_{n+1} \;=\; 0\ ,
\end{aligned}$$

respectively. A new approach to definite and indefinite summation problems is by the application of algorithmic techniques for summation. Two of several powerful methods are known as *Gosper's algorithm* for indefinite hypergeometric summation as well as *Zeilberger's algorithm* for definite hypergeometric summation. In [11] the up-to-date algorithmic techniques are described in detail and worked out using Maple programs. Particularly, in chapter 7 of the book Zeilberger's algorithm is introduced as an extension of Gosper's algorithm with which one can not only prove hypergeometric identities but also sum definite series in many cases, provided that they represent hypergeometric terms. The corresponding Maple procedure is printed out on page 100. In what follows, we apply Zeilberger's algorithm to prove some results, but also introduce a very simple numeric algorithm using only standard Maple commands, which allows to compute easily all the recurrence formulae given in this paper. First, let us consider the sums

$$u_{n,0} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^3$$

$$= {}_5F_4 \left( \begin{matrix} -n & -n & n+1 & n+1 & n+1 \\ 1 & 1 & 1 & 1 & \end{matrix} \,|\, 1 \right) , \tag{2}$$

$$u_{n,1} = \sum_{k=0}^{n} k \cdot \binom{n}{k}^2 \binom{n+k}{k}^3$$

$$= n^2(n+1)^3 \cdot {}_5F_4 \left( \begin{matrix} -n+1 & -n+1 & n+2 & n+2 & n+2 \\ 2 & 2 & 2 & 2 & \end{matrix} \,|\, 1 \right) . \tag{3}$$

On the one side we shall state a linear recurrence formula for (2) resp. (3), on the other side we prove a much simpler identity involving both sums. By Zeilberger's algorithm, we first get the following result.

**Theorem 1**: *For all positive integers $n$ the recurrence formulae*

$$P_1(n)u_{n-2,0} + P_2(n)u_{n-1,0} + P_3(n)u_{n,0} + P_4(n)u_{n+1,0} + P_5(n)u_{n+2,0} \;=\; 0$$

*and*

$$Q_1(n)u_{n-2,1} + Q_2(n)u_{n-1,1} + Q_3(n)u_{n,1} + Q_4(n)u_{n+1,1} + Q_5(n)u_{n+2,1} \;=\; 0$$

*hold for polynomials $P_i(n)$, $Q_i(n)$ for $i = 1, 2, 3, 4, 5$. Particularly, one has:*

$$P_1(n) =$$

$$n^2(n-1)^4(843719n^6 + 4840551n^5 + 11178979n^4 + 13317705n^3 + 8665902n^2 +$$
$$+2935212n + 406064) ,$$

$$P_2(n) =$$

$$n^2(36279917n^{10} + 135583859n^9 + 127047837n^8 - 60969738n^7 - 110574890n^6 +$$
$$+20533918n^5 + 48708406n^4 - 5364623n^3 - 11899162n^2 + 654644n + 1218192) ,$$

$$P_3(n) =$$

$$1736373702n^{12} + 9961853958n^{11} + 22277118313n^{10} + 23185704893n^9 + 8094752075n^8 -$$
$$-5159542775n^7 - 5379560162n^6 - 735829596n^5 + 854942060n^4 + 330715196n^3 -$$
$$-27939752n^2 - 33839440n - 4927520 ,$$

$$P_4(n) =$$

$$-(n+1)^2(109683470n^{10} + 629271630n^9 + 1276828039n^8 + 883221686n^7 - 311491995n^6 -$$

$$-543643850n^5+7337532n^4+118619820n^3+524952n^2-10038672n+143520)\ ,$$

$$P_5(n)=$$

$$(n+1)^2(n+2)^4(843719n^6-221763n^5-367991n^4+132919n^3+36936n^2-18952n+1196)\ ;$$

$$Q_1(n)=$$

$$n^3(n-1)^3(843719n^8+6997005n^7+25031348n^6+50399270n^5+62438227n^4+48758733n^3+$$

$$+23462050n^2+6361792n+744400)\ ,$$

$$Q_2(n)=$$

$$n^2(n+1)(36279917n^{11}+192031464n^{10}+325366682n^9+63867151n^8-330040841n^7-$$

$$-172445298n^6+193400596n^5+124585299n^4-57601426n^3-40293768n^2+6706960n+$$

$$+4883264)\ ,$$

$$Q_3(n)=$$

$$1736373702n^{14}+12663462588n^{13}+37617073049n^{12}+56520762812n^{11}+40640562631n^{10}+$$

$$+4324668130n^9-11874058437n^8-4827274624n^7+1559429331n^6+752392334n^5-$$

$$-460798228n^4-194231560n^3+70509632n^2+46662400n+6653440\ ,$$

$$Q_4(n)=$$

$$-(n+1)^2(109683470n^{12}+690243710n^{11}+1628605979n^{10}+1620805286n^9+286252536n^8-$$

$$-554888138n^7-210859869n^6+104854206n^5+36340108n^4-26323720n^3-7970880n^2+$$

$$+4292352n+1589760)\ ,$$

$$Q_5(n)=$$

$$(n+1)^3(n+2)^3(843719n^8+247253n^7-323555n^6-99977n^5+77252n^4+18476n^3-$$

$$-20240n^2-1472n+2944)\ .$$

Assuming five-term recurrence formulae for $u_{n,0}$ and $u_{n,1}$, one can compute the coefficients of the polynomials $P_i$ and $Q_i$, respectively, by the following simple Maple-procedure solving a linear quadratic system of equations. Here is the procedure for $u_{n,0}$:

```
> with(linalg):
> f := t -> sum(binomial(t,k)^2 * binomial(t+k,k)^3, k = 0..t):
> g := 12:
> p := 5 * (1 + g):
> h1 := (n,m) -> n + floor((m-1)/(1+g)):
> h2 := m -> -m mod(g+1):
> M := Matrix(p,p,(n,m) -> (n+2)^(h2(m)) * f(h1(n,m))):
> b := Vector[row](p, n -> 0):
> x := linsolve(M,b);
```

The coefficients $a_{i,k}$ of the polynomials

$$p_i(x) \;=\; \sum_{k=0}^{g} a_{i,k} x^{g-k} \qquad (i = 1,2,3,4,5)$$

are the unknowns of the homogeneous $(p \times p)$-system. The matrix of the system is given by

$$\big(n^g f(n-2), n^{g-1} f(n-2), \ldots, f(n-2); n^g f(n-1), n^{g-1} f(n-1), \ldots, f(n-1); n^g f(n), \ldots\;; \\ n^g f(n+2), \ldots, f(n+2)\big)_{n=3,4,\ldots,p+2}\;.$$

The output of the above Maple-procedure is a one-dimensional vector space in $\mathbb{R}^p$. The smallest nontrivial integer solution of this space (apart from its sign) gives the desired integer coefficients

$$(a_{1,0}, a_{1,1}, \ldots, a_{1,g}; a_{2,0}, \ldots, a_{2,g}; \ldots; a_{5,0}, \ldots, a_{5,g})^T\;.$$

Choosing $g$ smaller than 12, the system of equations has no nontrivial solution, which proves that there are no polynomials in the five-term recurrence formula for $u_{n,0}$ having a degree smaller than 12 each. Conversely, the existence of a nontrivial solution for $g \geq 12$ does *not* prove that the recurrence formula in Theorem 1 holds *for all* positive integers $n$. But this follows from Zeilberger's algorithm, which produces the same polynomials as the above Maple-procedure. To compute the coefficients of the polynomials $Q_i$ in Theorem 1, the algorithm works by setting

$$>\; f := t \to sum(k * binomial(t,k)\hat{}2 * binomial(t+k,k)\hat{}3, k = 0..t):$$

Put $g = 14$; there is no nontrivial solution for any smaller $g$.

By the result of the following theorem a much simpler linear equation is found involving both sums, $u_{n,0}$ and $u_{n,1}$:

**Theorem 2**:

$$\begin{aligned} n^3(216n+125)\,u_{n-1,0} \;-\; (3471n^4 + 6057n^3 + 4278n^2 + 1472n + 207)\,u_{n,0} + \\ + 23(n+1)^4\,u_{n+1,0} + \\ + 5n^2(76n+41)\,u_{n-1,1} \;+\; (1161n^3 + 1083n^2 + 207n - 23)\,u_{n,1} \;=\; 0 \end{aligned} \tag{4}$$

This five term recurrence does not follow automatically from Zeilberger's algorithm. The idea to find the coefficients of the polynomials is to modify the above Maple-procedure in the following way:

$$\begin{aligned}
&>\; e := && t \to sum(binomial(t,k)\hat{}2 * binomial(t+k,k)\hat{}3, k = 0..t):\\
&>\; f := && t \to sum(k * binomial(t,k)\hat{}2 * binomial(t+k,k)\hat{}3, k = 0..t):\\
&>\; r := && (m,g) \to floor(abs((m-1)/(2*(1+g)) - 1/2)):\\
&>\; g := && 4:\\
&>\; p := && 5*(1+g):\\
&>\; h1 := && (n,m) \to n + floor((m-1)/(1+g)):\\
&>\; h2 := && m \to -m \;\; mod(g+1):\\
&>\; M := && Matrix(p,p,(n,m) \to (n+4)\hat{}(h2(m)) * ((1-r(m,g)) * e(h1(n+3,m))\\
& && \quad + r(m,g) * f(h1(n,m)))):\\
&>\; b := && Vector[row](p, n \to 0):\\
&>\; x := && linsolve(M,b);
\end{aligned}$$

**Proof of Theorem 2:** For integers $k, n$ with $0 \le k \le n$, we put

$$\lambda_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^3 ,$$

where $\lambda_{n,k} = 0$ if $k < 0$ or $k > n$ .

$$\begin{aligned}
A_0 &= n^3(216n + 125)\,, \\
A_1 &= 5n^2(76n + 41)\,, \\
B_0 &= -\,(3471n^4 + 6057n^3 + 4278n^2 + 1472n + 207)\,, \\
B_1 &= 1161n^3 + 1083n^2 + 207n - 23\,, \\
C_0 &= 23(n+1)^4\,, \\
D_1 &= 4(144n^3 + 252n^2 + 138n + 23)\,, \\
D_2 &= 4(296n^2 + 288n + 69)\,, \\
L_{n,k} &= A_0 + B_0 + C_0 + D_1 k + D_2 k^2\,.
\end{aligned}$$

The proof of the theorem is based on the identity

$$\begin{aligned}
(A_0 + A_1 k)\,\lambda_{n-1,k} \;+\; (B_0 + B_1 k)\,\lambda_{n,k} \;+\; C_0 \lambda_{n+1,k} \\
= L_{n,k}\lambda_{n,k} \;-\; L_{n,k-1}\lambda_{n,k-1}
\end{aligned} \tag{5}$$

for $n \ge 1$ , $0 \le k \le n$ . Let $0 \le k \le n$. We divide (5) by $\lambda_{n,k}$; note that

$$\frac{\lambda_{n-1,k}}{\lambda_{n,k}} = \frac{n(n-k)^2}{(n+k)^3}\ , \quad \frac{\lambda_{n+1,k}}{\lambda_{n,k}} = \frac{(n+k+1)^3}{(n+1)(n-k+1)^2}\ ,$$

$$\frac{\lambda_{n,k-1}}{\lambda_{n,k}} = \frac{k^5}{(n-k+1)^2(n+k)^3}\ .$$

Putting these terms into (5), multiplying with $(n+1)(n+k)^3(n-k+1)^2$ and arranging the terms with respect to the powers of $k$, we get

$$\begin{aligned}
&\{(A_0 + B_0 + C_0)n^3(n+1)^3\} \\
&+\{-2A_0 n^2(n+1)(2n^2+3n+1) + B_0 n^2(n+1)(n^2+4n+3) \\
&\quad +3C_0 n^2(n+1)^2(2n+1) + (A_1+B_1)n^3(n+1)^3\}k \\
&+\{A_0 n(n+1)(6n^2+6n+1) - B_0 n(n+1)(2n^2-3) \\
&\quad +3C_0 n(n+1)(5n^2+5n+1) - 2A_1 n^2(n+1)(2n^2+3n+1) \\
&\quad +B_1 n^2(n+1)(n^2+4n+3)\}k^2 \\
&+\{-2A_0 n(n+1)(2n+1) - B_0(n+1)(2n^2+4n-1) \\
&\quad +C_0(2n+1)(10n^2+10n+1) + A_1 n(n+1)(6n^2+6n+1) \\
&\quad -B_1 n(n+1)(2n^2-3)\}k^3 \\
&+\{A_0 n(n+1) + B_0(n+1)(n-2) + 3C_0(5n^2+5n+1) \\
&\quad -2A_1 n(n+1)(2n+1) - B_1(n+1)(2n^2+4n-1)\}k^4 \\
&+\{B_0(n+1) + 3C_0(2n+1) + A_1 n(n+1) + B_1(n+1)(n-2)\}k^5 \\
&+\{C_0 + B_1(n+1)\}k^6
\end{aligned}$$

$$\begin{aligned}
&= \{(A_0+B_0+C_0)n^3(n+1)^3\} \\
&+\{D_1n^3(n+1)^3+(A_0+B_0+C_0)n^2(n+1)(n^2+4n+3)\}k \\
&+\{D_1n^2(n+1)(n^2+4n+3)+D_2n^3(n+1)^3 \\
&\quad -(A_0+B_0+C_0)n(n+1)(2n^2-3)\}k^2 \\
&+\{-D_1n(n+1)(2n^2-3)+D_2n^2(n+1)(n^2+4n+3) \\
&\quad -(A_0+B_0+C_0)(n+1)(2n^2+4n-1)\}k^3 \\
&+\{-D_1(n+1)(2n^2+4n-1)-D_2n(n+1)(2n^2-3) \\
&\quad +(A_0+B_0+C_0)(n+1)(n-2)\}k^4 \\
&+\{D_1(n+1)(n-2)-D_2(n+1)(2n^2+4n-1)+(A_0+B_0+C_0)(n+1)\}k^5 \\
&+\{D_1(n+1)+D_2(n+1)(n-2)\}k^6 \\
&+\{D_2(n+1)\}k^7 \\
&-\{-D_1(n+1)+D_2(n+1)+(A_0+B_0+C_0)(n+1)\}k^5 \\
&-\{D_1(n+1)-2D_2(n+1)\}k^6 \\
&-\{D_2(n+1)\}k^7
\end{aligned} \tag{6}$$

(6) may be considered as a polynomial in $k$ of degree 7. Therefore it suffices to treat the polynomials in $n$ belonging to the same power of $k$ by straightforward computations. This can be simplified if we put in for $n$ sufficiently many of the numbers $0,1,2,\dots$ to check the identities of the polynomials in $n$. — Hence (5) is proved. Finally, we sum up from $k=0$ to $n$ in (5). We have

$$\sum_{k=0}^{n}(L_{n,k}\lambda_{n,k} - L_{n,k-1}\lambda_{n,k-1}) = L_{n,n}\lambda_{n,n}$$

(by $\lambda_{n,-1}=0$) and

$$\sum_{k=0}^{n}\lambda_{n-1,k} = \sum_{k=0}^{n-1}\lambda_{n-1,k} = \mu_{n-1,0}$$

(by $\lambda_{n-1,n}=0$).
So one gets

$$\left(A_0\,\mu_{n-1,0}+B_0\,\mu_{n,0}+C_0\cdot\sum_{k=0}^{n}\lambda_{n+1,k}\right)+A_1\,\mu_{n-1,1}+B_1\,\mu_{n,1}=L_{n,n}\,\lambda_{n,n}\,.$$

To finish the proof of the theorem it suffices to show that

$$-C_0\,\lambda_{n+1,n+1} = L_{n,n}\,\lambda_{n,n}\ ,$$

which is equivalent with

$$8C_0(2n+1)^3 + (n+1)^3\left(A_0+B_0+C_0+D_1n+D_2n^2\right) = 0\ .$$

Here the identity

$$\frac{\lambda_{n+1,n+1}}{\lambda_{n,n}} = \left(\frac{2\cdot(2n+1)}{n+1}\right)^3$$

is used. The last but one identity can be verified by straightforward computations. The proof of the theorem is complete. $\square$

It has been already mentioned in the introduction that

$$v_{n,0} := \sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}$$

satisfies a five-term recurrence formula with integer polynomials of degree 8. Moreover, the same is true for the sum

$$v_{n,1} := \sum_{k=0}^{n} k \binom{n}{k}^3 \binom{n+k}{k}$$

with integer polynomials of degree 10. Similarly to the result of Theorem 2, one can prove a much simpler linear identity involving both sums, $v_{n,0}$ and $v_{n,1}$:

**Theorem 3**:

$$2n^2(343n+135)\, v_{n-1,0} + (412n^3+537n^2+272n+51)\, v_{n,0} -$$

$$-17(n+1)^3\, v_{n+1,0} +$$

$$+96n(8n+3)\, v_{n-1,1} - (162n^2+99n+17)\, v_{n,1} = 0.$$

The arguments in proving that result are the same as in the proof of Theorem 2. The coefficients of the polynomials can be found by a Maple-procedure adapted to $v_{n,0}$ and $v_{n,1}$.

## 2. EXPLICIT FORMULAE FOR THE LIMITS OF SOME SERIES WITH BINOMIAL COEFFICIENTS

It is of great interest that Apery's proof of the irrationality of $\zeta(3)$ leads to an infinite series consisting of terms associated with $\binom{2n}{n}^{-1}$, which converges rapidly to $\zeta(3)$:

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} \,. \tag{7}$$

A similar identity holds for $\zeta(2)$ :

$$\zeta(2) = \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} \,, \qquad ([3]). \tag{8}$$

Furthermore, it is known that

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{2\pi\sqrt{3}+9}{27} \,, \tag{9}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9} \,. \tag{10}$$

(8), (9), and (10) can be obtained from

$$2\,(\arcsin z)^2 = \sum_{n=1}^{\infty} \frac{(2z)^{2n}}{n^2\binom{2n}{n}}$$

by computing the first and second derivative at the point $z = \frac{1}{2}$ (see [15], [5]). The same arguments yield (at the point $z = \frac{i}{2}$) :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 \binom{2n}{n}} = 2 \log^2 \left( \frac{\sqrt{5}-1}{2} \right) , \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \binom{2n}{n}} = \frac{2\sqrt{5}}{5} \log \left( \frac{\sqrt{5}+1}{2} \right) , \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4\sqrt{5}}{25} \log \left( \frac{\sqrt{5}+1}{2} \right) . \tag{13}$$

(13) can be found in [10]. It was conjectured by *A. van der Poorten* that

$$\zeta(4) \left( = \frac{\pi^4}{90} \right) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} . \tag{14}$$

Finally this conjecture was proved by A. van der Poorten himself [14], [15] and by *H. Cohen* [4] . In the same paper [4] H. Cohen has generalized the ideas of R. Apery for $\zeta(2r+1)$ and $\zeta(2r)$ $(r \geq 1)$, which leads (unfortunately) to no new irrationality proofs, but to some new identities. For instance:

$$\zeta(5) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \binom{2n}{n}} \left( \sum_{k=1}^{n-1} \frac{1}{k^2} - \frac{4}{5n^2} \right) ; \tag{15}$$

for some related identities see [12].

In the following we state the limits of some related sums; all these identities can be proved by using the real (or complex) Taylor expansion of a certain function, or by taking advantage of the known value of a certain integral. For more results we refer the reader to the chapter on *numerical power series* in [10].

$$\sum_{n=0}^{\infty} \frac{n}{(2n+1)\binom{2n}{n}} = \frac{2}{3} - \frac{2\pi\sqrt{3}}{27} , \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n}{(2n+1)\binom{2n}{n}} = \frac{12\sqrt{5}}{25} \log \left( \frac{1+\sqrt{5}}{2} \right) - \frac{2}{5} , \tag{16}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \binom{2n}{n}} = \frac{\pi}{2} \log 3 - 4 \cdot \int_0^1 \frac{\arcsin w}{4-w^2} \, dw , \tag{17}$$

$$\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}} = \frac{\pi}{2} , \tag{18}$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{4^n}{(2n+1)\binom{2n}{n}} = \frac{\sqrt{2}}{4} \log \left( \frac{\sqrt{2}+1}{\sqrt{2}-1} \right) . \tag{19}$$

These identities are proved by the integral

$$\int_0^{\frac{\pi}{2}} \left(\frac{\cos t}{2}\right)^{2m+1} dt = \frac{1}{2(2m+1)\binom{2m}{m}} \qquad (m \geq 0)\ ,$$

(see [9,vol.2,331-30]), and by [9,vol.1,231-10b].

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)(2n+1)\,4^n} = \pi - 2\,, \tag{20}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\,4^n} = \frac{\pi}{2}\,, \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2\,4^n} = \frac{\pi}{2}\,\log 2\,, \tag{22}$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{\binom{2n}{n}}{(2n+1)\,4^n} = \log(1+\sqrt{2})\,. \tag{23}$$

Here we have applied the series

$$\arcsin z \ = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)\,4^n} \cdot z^{2n+1} \qquad (|z| < 1)\ ,$$

and *Abel's limit theorem*. For the proof of (22) one may use the tables in [9,vol.2,341-6a].

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)\binom{2n}{n}} = \frac{1}{2} - \frac{\pi\sqrt{3}}{18}\ , \tag{24}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)\binom{2n}{n}} = 1 - \frac{\pi\sqrt{3}}{6}\ . \tag{25}$$

For a proof of (25) it is useful to apply the Taylor expansion of the function $(\arcsin x)^2$. The identity then follows by computing the limit on the right side of the identity

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)\binom{2n}{n}} = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\frac{1}{2}} \left(\frac{\arcsin t}{t^2\sqrt{1-t^2}} - \frac{1}{t}\right) dt$$

$$= \lim_{\varepsilon\to 0} \left[\,-(\arcsin t)\cdot(\cot \arcsin t)\,\right]_{t=\varepsilon}^{t=\frac{1}{2}}\ .$$

The identity in (24) follows easily applying (10), (25), and $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$.

The series $S_1$ and $S_2$ from the following theorem involving binomial coefficients $\binom{2n}{n}$ depend on a parameter $k$. They satisfy a homogeneous three-term recurrence formula each. Applications of the identities *(i)* and *(iii)* are given in Theorems 5,6 below and they provide a connection between the numbers $\sqrt{3}\pi$ and $\sqrt{5}\log\rho$ with $\rho := (1+\sqrt{5})/2$.

**Theorem 4**: *For every integer $k \ge 0$ we have*

$$(i) \qquad S_1(k) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+3)\cdot\ldots\cdot(2n+2k+1)\binom{2n}{n}} = \ldots$$

$$= \frac{4}{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)} \left( (-1)^k \frac{\pi}{2} 3^{k-\frac{3}{2}} + \sum_{\nu=0}^{k-1} \frac{(-3)^{\nu}}{2k-2\nu-1} \right)$$

$$(ii) \qquad (2k+3)^2 \cdot S_1(k+2) + 4(k+2)\cdot S_1(k+1) - 3\cdot S_1(k) = 0$$

$$(iii) \quad S_2(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)\cdot\ldots\cdot(2n+2k+1)\binom{2n}{n}} = \ldots$$

$$= \frac{4}{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)} \left( 5^{k-\frac{1}{2}} \log\left(\frac{1+\sqrt{5}}{2}\right) - \sum_{\nu=0}^{k-1} \frac{5^{\nu}}{2k-2\nu-1} \right)$$

$$(iv) \quad (2k+3)^2 \cdot S_2(k+2) - 4(3k+4)\cdot S_2(k+1) + 5\cdot S_2(k) = 0\,.$$

For instance, we get for $k = 1,2,3$:

$$S_1(1) = 4 - \frac{2\pi\sqrt{3}}{3}\,, \quad S_1(2) = \frac{2\pi\sqrt{3}}{3} - \frac{32}{9}\,, \quad S_1(3) = \frac{164}{75} - \frac{2\pi\sqrt{3}}{5}\,;$$

$$S_2(1) = 4\sqrt{5}\log\big(\frac{1+\sqrt{5}}{2}\big) - 4\,, \quad S_2(2) = \frac{20\sqrt{5}}{3}\log\big(\frac{1+\sqrt{5}}{2}\big) - \frac{64}{9}$$

$$S_2(3) = \frac{20\sqrt{5}}{3}\log\big(\frac{1+\sqrt{5}}{2}\big) - \frac{1612}{225}\,.$$

In the case when $k$ equals to zero we define as usual $\sum_{\nu=0}^{-1}(\ldots)$ to be 0, and $1\cdot 3\cdot\ldots\cdot(2k-1)$ to be 1 :

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\binom{2n}{n}} = \frac{2\pi\sqrt{3}}{9}\,, \tag{26}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\binom{2n}{n}} = \frac{4\sqrt{5}}{5}\log\left(\frac{1+\sqrt{5}}{2}\right)\,. \tag{27}$$

**Proof:** The linear three-term recurrences for $S_1(k)$ and $S_2(k)$ from this theorem are homogeneous ones. We apply Zeilberger's algorithm to both sums. Then one gets two inhomogeneous first order recurrences:

$$3S_1(k)+(2k+1)S_1(k+1)=4\cdot\frac{2^k k!}{(2k+1)!}\ ,\quad 5S_2(k)-(2k+1)S_2(k+1)=4\cdot\frac{2^k k!}{(2k+1)!}\ .\quad (28)$$

It can easily be seen that the terms on the right sides in *(i)* and *(iii)* satisfy the corresponding recurrence in (28). We demonstrate this for $S_1(k)$. For the sake of brevity let

$$a_k \ :=\ \sum_{\nu=0}^{k-1}\frac{(-3)^\nu}{2k-2\nu-1}\qquad (k\geq 1)\ .$$

One easily verifies that

$$a_{k+1}\ =\ -3a_k+\frac{1}{2k+1}\ .$$

We denote the term on the right side in *(i)* by $S_1^*(k)$. Then we have

$$\frac{2k+1}{3}\cdot S_1^*(k+1)\ =$$

$$=\ \frac{4}{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)}\cdot\left(-(-1)^k\cdot\frac{\pi}{2}\cdot 3^{k-\frac{3}{2}}-a_k+\frac{1}{3(2k+1)}\right)\ ,$$

or

$$\frac{2k+1}{3}\cdot S_1^*(k+1)+S_1^*(k)\ =\ \frac{4}{3\cdot(1\cdot 3\cdot 5\cdot\ldots\cdot(2k+1))}\ ,\qquad (29)$$

which gives the left recurrence formula in (28). To prove the identity in *(i)* it suffices to show that it holds for $k=0$, since both sides in *(i)* satisfy the same linear first order recurrence formula. So it remains to prove (26). For this purpose we apply the results from (9) and (16). By

$$\frac{1}{2}-\frac{n}{2n+1}\ =\ \frac{1}{2(2n+1)}\qquad (n\geq 0)$$

we get

$$\frac{S_1(0)}{2}=\sum_{n=0}^{\infty}\frac{1}{2(2n+1)\binom{2n}{n}}\ =\ \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}}-\sum_{n=0}^{\infty}\frac{n}{(2n+1)\binom{2n}{n}}$$

$$=\ \left(\frac{2}{3}+\frac{\pi\sqrt{3}}{27}\right)-\left(\frac{2}{3}-\frac{2\pi\sqrt{3}}{27}\right)=\frac{\pi\sqrt{3}}{9}\ ,$$

which gives (26). By similar arguments one shows that the identity in *(iii)* holds by using (13) and (16).

It can easily be seen that the three-term recurrence in *(ii)* follows from the first order recurrence in (29). Writing down this equation a second time with $k$ replaced by $k+1$ and then multiplying by $(2k+3)$, one gets an identity having the same right side as above. So a linear equation with respect to $S_1^*(k)$, $S_1^*(k+1)$, and $S_1^*(k+2)$ results which proves the identity from *(ii)* by $S_1^*(k) = S_1(k)$. The recurrence in *(iv)* can be proved analogously. This finishes the proof of the theorem. $\square$

The identities in *(i)* and *(iii)* also follow from the integral

$$\int_0^{\frac{\pi}{2}} \sin^{2k}t \cdot \cos^{2n+1}t\,dt = \frac{2^n \cdot n!}{(2k+1)(2k+3)\cdot \ldots \cdot (2k+2n+1)}$$

$$= \frac{2^{2n}\cdot z(k)}{(2n+1)(2n+3)\cdot \ldots \cdot (2n+2k+1)\binom{2n}{n}},$$

where $z(k) := 1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot (2k-1)$ $(k \geq 1)$, $z(0) := 1$ (see [9,vol.2,331-21c]). We leave the details to the reader.

Finally we show by the preceding results that the ordinary hypergeometric series provides the announced close connection between the numbers $\sqrt{3}\pi$ and $\sqrt{5}\log\rho$ with $\rho := (1+\sqrt{5})/2$. Putting

$$c_m := \frac{(-1)^m \cdot m!}{4^m} \qquad (m \geq 1)\ ,$$

one easily checkes by straightforward calculations that

$$S_2(n-1) = \frac{2^{n-1}\cdot (n-1)!}{(2n-1)!}\cdot \sum_{m=0}^{\infty} \frac{c_m}{(n+\frac{1}{2})_m} \tag{30}$$

$$= \frac{2^{n-1}\cdot (n-1)!}{(2n-1)!}\cdot F\left(1,1;n+\frac{1}{2};-\frac{1}{4}\right)$$

holds for all integers $n \geq 1$. Here,

$$F(a,b;c;x) = {}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix} \,\middle|\, x\right)$$

denotes the ordinary hypergeometric function. A similar identity can be found for $S_1$. So we have proved the following result.

**Theorem 5**: *For every positive integer $n$ there are rationals $q, r, s, t$ such that*

$$F\left(1,1;n+\frac{1}{2};\frac{1}{4}\right) = q\sqrt{3}\pi + r$$

*and*

$$F\left(1,1;n+\frac{1}{2};-\frac{1}{4}\right) = s\sqrt{5}\rho + t$$

*hold.*

But two more general identities can be proved. Let

$$s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \qquad (n \geq 2) .$$

One has

$$s_n = \gamma + O(n^{-1}) \qquad (n \geq 2) ,$$

where $\gamma$ denotes Euler's constant. In [6] the author has shown that

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n+k+\tau-1}{n} \binom{n}{k} \cdot s_{k+\tau} = \gamma + O\left(\frac{(\tau-1)!}{(n)_{\tau+1}}\right)$$

holds for all integers $n \geq 1$ and $\tau \geq 2$. The specific linear form in $s_\tau, \ldots, s_{n+\tau}$ with integer coefficients converges more rapidly to $\gamma$ than the basic series $s_n$. The underlying idea of that series transformation goes back to an identity proved by J. Ser in 1925 [17]. The form of Ser's result the author has used in [6] is the following one:

$$s_{n+\tau} = \gamma - \sum_{m=1}^{\infty} \frac{c_m}{(n+\tau)_m} \qquad (n \geq 0)$$

with explicitly given rationals $c_m$. As can be seen by (30), a similar series transformation can be applied to $S_2$ (and to $S_1$, of course). Finally, one gets:

**Theorem 6**: *Let $n$ and $m$ denote positive integers satisfying $m \geq 2n$. Then there are rationals $q, r, s, t$ such that*

$$F\left(n, n+1; m+\frac{1}{2}; \frac{1}{4}\right) = q\sqrt{3}\pi + r$$

*and*

$$F\left(n, n+1; m+\frac{1}{2}; -\frac{1}{4}\right) = s\sqrt{5}\rho + t$$

*hold.*

For instance, we have for $n = 5$ and $m = 10$:

$$F\left(5, 6; \frac{21}{2}; \frac{1}{4}\right) = 4676174360\sqrt{3}\pi - \frac{178114483690}{7} ,$$

$$F\left(5, 6; \frac{21}{2}; -\frac{1}{4}\right) = -55464490224\sqrt{5}\log\rho + \frac{417767218726}{7} .$$

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