FIBONACCI REPRESENTATIONS

L. CARLITZ
Duke University, Durham, North Carolina

1. INTRODUCTION

We define the Fibonacci numbers as usual by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

We shall be concerned with the problem of determining the number of representations of a given positive integer as a sum of distinct Fibonacci numbers. More precisely we define $R(N)$ as the number of representations

(1.1) $$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$

where

(1.2) $$k_1 > k_2 > \cdots > k_r \geq 2 ;$$

the integer r is allowed to vary. We shall refer to (1.1) as a Fibonacci representation of N provided (1.2) is satisfied.

This definition is equivalent to

(1.3) $$\prod_{n=2}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R(N)\, y^N$$

with $R(0) = 1$. We remark that Hoggatt and Basin [4] have discussed a closely related function defined by

(1.4) $$\prod_{n=1}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R'(N)\, y^N .$$

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Comparing (1.4) with (1.3) it is evident that

$$R'(N) = R(N) + R(N-1) \quad . \tag{1.5}$$

Ferns [3] and Klarner [5] have also discussed the problem of representing an integer as a sum of distinct Fibonacci numbers. We recall that by a theorem of Zeckendorf [1] the representation (1.1) is unique provided the k_j satisfy the inequalities

$$\cdots\ k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r-1); \quad k_r \geq 2 \quad . \tag{1.6}$$

We call such a representation the <u>canonical</u> representation of N.

Rather than work directly with $R(N)$ we shall find it convenient to define the function $A(m,n)$ by means of

$$\prod_{n=1}^{\infty} (1 + x^{F_n} y^{F_{n+1}}) = \sum_{m,n=0}^{\infty} A(m,n) x^m y^n \quad . \tag{1.7}$$

It is easily seen that $A(m,n)$ satisfies the recurrence

$$A(m,n) = A(n-m,n) + A(n-m, m-1) \quad . \tag{1.8}$$

Also, as we shall see,

$$R(N) = A(e(N), N) \quad , \tag{1.9}$$

where

$$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_r-1} \quad , \tag{1.10}$$

and the k_s are determined by (1.1); the value of $e(N)$ is independent of the particular Fibonacci representation employed. In particular we may assume that the representation (1.1) is canonical. Indeed most of the theorems of the paper make use of the canonical representation.

In particular it follows from (1.9) that for fixed n there is a unique value of m, namely $e(n)$, such that $A(m,n) \neq 0$.

It is helpful to make a short list of exponent pairs occurring in the right member of (1.7), that is, pairs (m,n) such that $A(m,n) \neq 0$. Using the recurrence (1.8) we get the following:

$$
\begin{array}{l}
1\ 1,\quad 1\ 2 \mid 2\ 3 \mid 3\ 4,\quad 3\ 5 \mid 4\ 6,\quad 4\ 7 \mid 5\ 8 \mid \\
6\ 9,\quad 6\ 10 \mid 7\ 11 \mid 8\ 12,\quad 8\ 13 \mid 9\ 14,\quad 9\ 15 \mid 10\ 16 \mid \\
11\ 17,\quad 11\ 18 \mid 12\ 19,\quad 12\ 20 \mid 13\ 21 \mid 14\ 22,\quad 14\ 23 \mid 15\ 24 \mid \\
\cdots\ 16\ 25,\quad 16\ 26 \mid 17\ 27,\quad 17\ 28 \mid 18\ 29 \mid 19\ 30,\quad 19\ 31 \mid 30\ 32 \mid \\
21\ 33,\quad 21\ 34 \mid 22\ 35,\quad 22\ 36 \mid 23\ 37 \mid 24\ 38,\quad 24\ 39 \mid \\
25\ 40,\quad 25\ 41 \mid 26\ 42 \mid 27\ 43,\quad 27\ 44 \mid 28\ 45 \mid
\end{array}
$$

This suggests that for given n, there are just one or two values of m such that $A(m,n) \neq 0$. As we shall see, this is indeed the case.

The first main result of the paper is a reduction formula (Theorem 1) which theoretically enables one to evaluate $R(N)$ for arbitrary N. While explicit formulas are obtained for $r = 1, 2, 3$ in a canonical representation, the general case is very complicated. If, however, we assume that all the k_s have the same parity the situation is much more favorable. Indeed if we assume that

$$N = F_{2k_1} + \cdots + F_{2k_r} \qquad (k_1 > \cdots > k_r \geq 1)$$

and put

$$j_s = k_s - k_{s+1} \qquad (s = 1, \cdots, r-1); \; j_r = k_r \quad ,$$

$$f_r = f(j_1, \cdots, j_r) = R(N), \quad S_r = 1 + f_1 + f_2 + \cdots + f_r \quad ,$$

then we have

$$S_0 = 1, \quad S_1 = j_1 + 1, \quad S_r = (j_r + 1)S_{r-1} - S_{r-2} \qquad (r \geq 2)$$

In particular if $j_1 = \cdots = j_r = j$ we have

$$S_r = \sum_{2t \le r} (-1)^t \binom{r-t}{t} (j+1)^{r-2t}$$

Returning to (1.10) we show also that if $k_r > 2$, then $e(N) = \{\alpha^{-1}N\}$, the integer nearest to $\alpha^{-1}N$, where $\alpha = (1 + \sqrt{5})/2$, while for $k_r = 2$, $e(N) = [\alpha^{-1}N] + 1$.

Additional applications of the method developed in this paper will appear later.

Section 2

As noted above, by the theorem of Zeckendorf, the positive N possesses a unique representation

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} , \tag{2.1}$$

with

$$k_j - k_{j+1} \ge 2 \quad (j = 1, \cdots, r-1); \quad k_r \ge 2 . \tag{2.2}$$

When (2.2) is satisfied we shall call (2.1) the canonical representation of N. Then the set of integers $(k_1, k_2, \cdots, k_r)$ is uniquely determined by N and conversely.

The following lemma will be required.

Lemma. Let

$$N = F_{k_1} + \cdots + F_{k_r} = F_{j_1} + \cdots + F_{j_s} , \tag{2.3}$$

where

$$k_1 > k_2 > \cdots > k_r \ge 2 ; \quad j_1 > j_2 > \cdots > j_s \ge 2 \tag{2.4}$$

be any two Fibonacci representations of N. Then

(2.5) $$F_{k_1-1} + \cdots + F_{k_r-1} = F_{j_1-1} + \cdots + F_{j_s-1} .$$

Proof. The lemma obviously holds for $N = 1$. We assume that it holds up to and including the value $N - 1$. If $k_1 = j_1$ then (2.3) implies

$$F_{k_2} + \cdots + F_{k_r} = F_{j_2} + \cdots + F_{j_s} < N$$

and (2.5) is an immediate consequence of the inductive hypothesis. We may accordingly assume that $k_1 > j_1$. Since

$$\cdots F_2 + F_3 + \cdots + F_n = F_{n+2} - 2 ,$$

we must have $k_1 = j_1 + 1$. If $k_2 = k_1 - 1$ we can complete the induction as in the previous case. If $k_2 = k_1 - 2$, (2.3) implies

(2.6) $$2F_{k_2} + F_{k_3} + \cdots + F_{k_r} = F_{j_2} + \cdots + F_{j_s} ,$$

with $j_2 \leq k_2$. If $j_2 < k_2$,

$$F_{j_2} + \cdots + F_{j_s} \leq F_2 + F_3 + \cdots + F_{k_2-1} < F_{k_2+1} < 2F_{k_2} ,$$

which contradicts (2.6). If $j_2 = k_2$, (2.6) reduces to

$$F_{k_2} + F_{k_3} + \cdots + F_{kr} = F_{j_3} + \cdots + F_{j_s} < N .$$

Then by the inductive hypothesis

(2.7) $$F_{k_2-1} + F_{k_3-1} + \cdots + F_{k_r-1} = F_{j_3-1} + \cdots + F_{j_s-1} .$$

Since $j_1 = k_1 - 1$, $j_2 = k_2 = k_1 - 2$, we have

$$F_{k_1-1} = F_{j_1} = F_{j_1-1} + F_{j_1-2} = F_{j_1-1} + F_{j_2-1} ,$$

so that (2.7) implies (2.5).

Finally there is the possibility $F_{k_2} < F_{k_1} - 2$. In this case (2.3) reduces to

$$(2.8) \qquad F_{k_1-2} + F_{k_2} + \cdots + F_{k_r} = F_{j_2} + \cdots + F_{j_s} = N' < N ;$$

each member of (2.8) is a Fibonacci representation of N'. By the inductive hypothesis

$$(2.9) \qquad \cdots F_{k_1-3} + F_{k_2-1} + \cdots + F_{k_r-1} = F_{j_2-1} + \cdots + F_{j_s-1} .$$

Since $j_1 - 1 = k_1 - 2$, (2.9) implies

$$F_{k_1} + F_{k_2-1} + \cdots + F_{k_r-1} = F_{j_1-1} + F_{j_2-1} + \cdots + F_{j_s-1}$$

and the induction is complete.

This evidently completes the proof of the lemma.

We now make the following

Definition. Let

$$(2.10) \qquad N = F_{k_1} + \cdots + F_{k_r} \qquad (k_1 > \cdots > k_r \geq 2)$$

be any Fibonacci representation of the positive integer N. Then we define

$$(2.11) \qquad e(N) = F_{k_1-1} + \cdots + F_{k_r-1} .$$

It is convenient to define

$$(2.12) \qquad e(0) = 0 .$$

In view of the lemma it is immaterial which Fibonacci representation of N we use in defining $e(N)$. In particular we may use the canonical representation (2.1).

Section 3

Returning to (1.7) we put

$$(3.1) \qquad \Phi(x, y) = \prod_{n=1}^{\infty} (1 + x^{F_n} y^{F_{n+1}}) .$$

Then

$$\Phi(x,xy) = \prod_{n=1}^{\infty} (1 + x^{F_n+F_{n+1}}) = \prod_{n=2}^{\infty} (1 + y^{F_n}x^{F_{n+1}}) ,$$

$$\cdots$$

so that

$$(1 + xy)\Phi(x,xy) = \Phi(y,x) .$$

Hence

$$(1 + xy) \sum_{m,n=0}^{\infty} A(m,n)x^{m+n}y^n = \sum_{m,n=0}^{\infty} A(m,n)y^m x^n .$$

Comparison of coefficients yields

$$A(m,n) = A(n - m, m) + A(n - m, m - 1) , \tag{3.2}$$

the recurrence stated in the Introduction.

In the next place it is clear from the definition of $e(N)$ that (1.3) reduces to

$$\prod_{n=1}^{\infty} (1 + x^{F_n}y^{F_{n+1}}) = \sum_{N=0}^{\infty} R(N)\, x^{e(N)}y^N , \tag{3.3}$$

where $R(N)$ is defined by

$$\prod_{n=2}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R(N)y^N . \tag{3.4}$$

It follows that

$$R(N) = A(e(N), N) \ . \tag{3.5}$$

In particular we see that, for fixed n, there is a unique value of m, namely $e(n)$, such that $A(m,n) \neq 0$.

If we take $m = e(n)$ in (3.2) we get

$$R(N) = A(N - e(N),\ e(N)) + A(N - e(N), e(N) - 1) \ . \tag{3.6}$$

Now let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} \tag{3.7}$$

with k_r odd. Then

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} \ ,$$

$$N - e(N) = F_{k_1-2} + \cdots + F_{k_r-2} \ .$$

Since $k_r \geq 3$, it follows that

$$N - e(N) = e(e(N)) \ . \tag{3.8}$$

On the other hand, since

$$F_3 + F_5 + \cdots + F_{2t-1} = F_{2t} - 1 \ ,$$

we have, for $k_r = 2t + 1$,

$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{r-1}} + (F_3 + F_5 + \cdots + F_{2t-1}) \ ;$$

the right member is evidently a Fibonacci representation, so that

$$\begin{aligned} e(e(N) - 1) &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + (F_2 + F_4 + \cdots + F_{2t-2}) \\ &= F_{k_1-2} + \cdots + F_{k_{r-1}-2} + F_{k_r-2} - 1 \\ &= N - e(N) - 1 \ . \end{aligned}$$

Thus

$$A(N - e(N), e(N) - 1) = 0$$

and (3.6) becomes

$$R(N) = A(e(e(N)), e(N)) .$$

In view of (3.8) we have

$$R(N) = R(e(N)) \qquad (k_r \text{ odd}) . \tag{3.9}$$

Now let k_r in the canonical representation of N be even. We shall show that

$$R(N) = R(e^{2t-1}(N_1)) + (t-1)R(e^{2t-2}(N_1)) , \tag{3.10}$$

where $k_r = 2t$,

$$N_1 = F_{k_1} + \cdots + F_{k_{r-1}} \tag{3.11}$$

and

$$e^t(N) = e(e^{t-1}(N)), \quad e^0(N) = N . \tag{3.12}$$

To prove (3.10) we take the canonical representation (3.7) with $k_r = 2t$. Then

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} , \tag{3.13}$$

which is a Fibonacci representation of e(N) except when t = 1. Excluding this case for the moment, we have as above

$$N - e(N) = e(e(N)) . \tag{3.14}$$

Moreover

$$
\begin{aligned}
e(N) - 1 &= F_{k_1 - 1} + \cdots + F_{k_{r-1}-1} + F_{2t-1} - 1 \\
&= F_{k_1 - 1} + \cdots + F_{k_{r-1}-1} + (F_2 + F_4 + \cdots + F_{2t-2}) \ , \\
e(e(N) - 1) &= F_{k_1 - 2} + \cdots + F_{k_{r-1}-2} + (F_1 + F_3 + \cdots + F_{2t-3}) \\
\cdots \qquad &= F_{k_1 - 2} + \cdots + F_{k_{r-1}-2} + F_{2t-2} \ ,
\end{aligned}
$$

so that

$$
e(e(N) - 1) = e(e(N)) \ . \tag{3.15}
$$

Substituting from (3.14) and (3.15) in (3.6) we get

$$
R(N) = R(e(N)) + R(e(N) - 1) \quad (k_r = 2t > 2) \ . \tag{3.16}
$$

When $k_r = 2$, (3.13) gives

$$
N - e(N) = F_{k_1 - 2} + \cdots + F_{k_{r-1}-2} = e(e(N_1)) \ ,
$$

$$
e(N) - 1 = F_{k_1 - 1} + \cdots + F_{k_{r-1}-1} = e(N_1) \ .
$$

Also since

$$
e(N) = F_{k_1 - 1} + \cdots + F_{k_{r-1}-1} + F_2 \ ,
$$

we get

$$
\begin{aligned}
e(e(N)) &= F_{k_1 - 2} + \cdots + F_{k_{r-1}-2} + F_1 \\
&= N - e(N) + 1 \ .
\end{aligned}
$$

It therefore follows from (3.5) and (3.6) that

$$
R(N) = R(e(N_1)) \qquad (k_R = 2) \ , \tag{3.17}
$$

in agreement with (3.10).

Returning to (3.16) we have first

$$(3.18) \qquad R(e(N)) = R(e^2(N)) \qquad (k_r = 2t > 2) ,$$

by (3.9). Since

$$e(N) - 1 = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + (F_2 + F_4 + \cdots + F_{2t-2}) ,$$

it follows by repeated application of (3.17) and (3.9) that

$$\begin{aligned} \cdots R(e(N) - 1) &= R(F_{k_1-2} + \cdots + F_{k_{r-1}-2} + F_3 + \cdots + F_{2t-3}) \\ &= R(F_{k_1-3} + \cdots + F_{k_{r-1}-3} + F_2 + \cdots + F_{2t-4}) \\ &= R(F_{k_1-2t+2} + \cdots + F_{k_{r-1}-2t+2}) \\ &= R(e^{2t-2}(N_1)) . \end{aligned}$$

Thus (3.16) becomes

$$(3.19) \qquad R(N) = R(e^2(N)) + R(e^{2t-2}(N_1)) \qquad (t > 1) .$$

Repeated use of (3.19) gives

$$R(N) = R(e^{2t-2}(N)) +)t - 1)R(e^{2t-2}(N_1)) ;$$

finally, applying (3.17), we get (3.10) .

Combining (3.9) and (3.10) we state the following principal result.

Theorem 1. Let N have the cannonical representation

$$N = F_{k_1} + \cdots + F_{k_r} ,$$

where

$$k_j - k_{j+1} \geq 2 \qquad (j = 1, \cdots, r - 1); \; k_r \geq 2 .$$

Then

$$R(N) = R(e^{k_r - 1}(N_1)) + ([\tfrac{1}{2}k_r] - 1)R(e^{k_r - 2}(N_1)), \tag{3.20}$$

where

$$N_1 = F_{k_1} + \cdots + F_{k_{r-1}} .$$

Section 4

Since

$$F_2 + F_4 + \cdots + F_{2t} = F_{2t+1} - 1, \quad F_1 + F_3 + \cdots + F_{2t-1} = F_{2t} ,$$

it follows that

$$e(F_{2t+1} - 1) = F_{2t}, \quad e(F_{2t} - 1) = F_{2t-1} - 1 . \tag{4.1}$$

Also since

$$F_{2t+1} - 2 = F_4 + F_6 + \cdots + F_{2t} ,$$

$$F_{2t} - 2 = F_2 + F_5 + F_7 + \cdots + F_{2t-1} ,$$

we get

$$e(F_{2t+1} - 2) = F_{2t} - 1, \quad e(F_{2t} - 2) = F_{2t-1} - 1 . \tag{4.2}$$

Now by (3.6), for $k \geq 2$,

$$\begin{aligned} R(F_k) &= A(F_{k-2}, F_{k-1}) + A(F_{k-2}, F_{k-1} - 1) \\ &= R(F_{k-1}) + A(F_{k-2}, F_{k-1} - 1) , \end{aligned}$$

$$\begin{aligned} R(F_k - 1) = A(F_k - 1 - e(F_k - 1), e(F_k - 1)) + A(F_k - 1 - e(F_k - 1), \\ e(F_k - 1) - 1). \end{aligned}$$

Then by (4.1),

$$A(F_{2t-2},\ F_{2t-1} - 1) = R(F_{2t-1} - 1),\quad A(F_{2t-1},\ F_{2t} - 1) = 0\ ,$$

so that

$$R(F_{2t}) = R(F_{2t-1}) + R(F_{2t-1} - 1),\quad R(F_{2t-1}) = R(F_{2t-2})\ . \tag{4.3}$$

In the next place by (4.1) and (4.3)

$$\begin{aligned} R(F_{2t} - 1) &= A(F_{2t-2}, F_{2t-1} - 1) + A(F_{2t-2}, F_{2t-1} - 2) \\ &= R(F_{2t-1} - 1)\ , \end{aligned}$$

$$\begin{aligned} R(F_{2t-1} - 1) &= A(F_{2t-3} - 1,\ F_{2t-2}) + A(F_{2t-3} - 1,\ F_{2t-2} - 1) \\ &= R(F_{2t-2} - 1). \end{aligned}$$

Hence we have

$$R(F_k - 1) = R(F_{k-1} - 1) \qquad (k \geq 2)\ , \tag{4.4}$$

which yields

$$R(F_k - 1) = 1 \qquad (k \geq 2)\ . \tag{4.5}$$

Substituting from (4.5) in (4.3), we get

$$R(F_{2t}) = R(F_{2t-1}) + 1,\quad R(F_{2t-1}) = R(F_{2t-2})\ ,$$

which implies

$$R(F_{2t}) = R(F_{2t+1}) = t \qquad (t \geq 1)\ . \tag{4.6}$$

We shall now show that $R(N) = 1$ implies $N = F_k - 1$. Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r}\ .$$

Then by (3.20)

$$R(e^{k_r - 1}(N_1)) = 1 \tag{4.7}$$

and $[k_r/2] = 1$, so that $k_r = 2$ or 3. Since

$$e^{k_r - 1}(N_1) = F_{k_1 - k_r + 1} + \cdots + F_{k_{r-1} - k_r + 1} ,$$

it is necessary that

$$[(k_{r-1} - k_r + 1)/2] = 1$$

and therefore

$$k_{r-1} - k_r = 2 .$$

Similarly

$$k_j - k_{j-1} = 2 \qquad (j = 1, 2, \cdots, r - 2) .$$

Hence we have either

$$N = F_{2r} + F_{2r-2} + \cdots + F_2 = F_{2r+1} - 1$$

or

$$N = F_{2r+1} + F_{2r-1} + \cdots + F_3 = F_{2r+2} - 1 .$$

We may sum up the results just obtained in the following theorems.

Theorem 2. We have

$$R(F_k) = [\tfrac{1}{2} k] \quad (k \geq 2) \tag{4.8}$$

Theorem 3. $R(N) = 1$ if and only if

$$N = F_k - 1, \quad k \geq 1 .$$

If we define $R'(N)$ by means of

$$(4.9) \qquad \prod_{n=1}^{\infty} (1 + y^{F_n}) = \sum_{N=0}^{\infty} R'(N) y^N ,$$

then

$$(4.10) \qquad R'(N) = R(N) + R(N - 1)$$

and it follows immediately that

$$(4.11) \qquad R'(F_k) = [\tfrac{1}{2} k] + 1 \qquad (k \geq 2) .$$

This result has been proved by Hoggatt and Basin [4].

Further results like (4.5) and (4.8) can be obtained by the same method. For example we can show that

$$R(F_{2t+1} - 2) = 1 + R(F_{2t} - 2) \qquad (t > 1) ,$$

$$R(F_{2t} - 2) = R(F_{2t-1} - 2) \qquad (t > 1) .$$

It follows that

$$(4.12) \qquad R(F_k - 2) = [\tfrac{1}{2} (k - 1)] \qquad (k \geq 3) .$$

Consequently by (4.11) we have

$$(4.13) \qquad R'(F_k - 1) = [\tfrac{1}{2}(k + 1)] ,$$

a result proved by Klarner [5, Th. 1].

Section 5

Theorem 1 furnishes a reduction formula by means of which $R(N)$ can be computed by arbitrary N. For example if

$$N = F_j + F_k \qquad (j - k \geq 2, \quad k \geq 2) \tag{5.1}$$

that by (3.20)

$$\begin{aligned} R(N) &= R(e^{k-1}(F_j)) + ([\tfrac{1}{2}k] - 1)R(e^{k-2}(F_j)) \\ &= R(F_{j-k+1}) + ([\tfrac{1}{2}k] - 1)\, R(F_{j-k+2}) \,. \end{aligned}$$

Applying (4.8) we get

$$R(N) = [\tfrac{1}{2}(j - k + 1)] + ([\tfrac{1}{2}k] - 1)[\tfrac{1}{2}(j - k + 2)] \,. \tag{5.2}$$

Again if

$$N = F_i + F_j + F_k \quad (i - j \geq 2,\; j - k \geq 2, \quad k \geq 2)\,, \tag{5.3}$$

then

$$R(N) = R(F_{i-k+1} + F_{j-k+1}) + ([\tfrac{1}{2}k] - 1)R(F_{i-k+2} + F_{j-k+2}) \,.$$

Applying (5.2) we get

$$\begin{aligned} R(N) &= [\tfrac{1}{2}(i - j + 1)] + ([\tfrac{1}{2}(j - k + 1)] - 1)\,[\tfrac{1}{2}(i - j + 2)] \\ &\quad + ([\tfrac{1}{2}k] - 1)\{[\tfrac{1}{2}(i - j + 1)] + (\tfrac{1}{2}[j - k + 2] - 1)[\tfrac{1}{2}(i - j + 2)]\} \,. \end{aligned} \tag{5.4}$$

Unfortunately, for general N the final result is very complicated. However (5.2) and (5.4) contain numerous special cases of interest.

In the first place, taking $k = 2, 3, 4$ in (5.2), we get

$$R(F_j + 1) = [\tfrac{1}{2}(j - 1)] \qquad (j \geq 4) \tag{5.5}$$

$$R(F_j + 2) = [\tfrac{1}{2}(j - 2)] \qquad (j \geq 5) \tag{5.6}$$

$$R(F_j + 3) = [\tfrac{1}{2}(j - 3)] + [\tfrac{1}{2}(j - 2)] \quad (j \geq 6) \,. \tag{5.7}$$

In the next place for the Lucas number L_k defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{k+1} = L_k + L_{k-1} \qquad (k \geq 1) ,$$

since $L_k = F_{k+1} + F_{k-1}$, (5.2) gives

$$R(L_k) = 1 + 2[\tfrac{1}{2}(k-3)] \qquad (k \geq 3) .$$

Hence

$$(5.8) \qquad \begin{aligned} R(L_{2k+1}) &= 2k - 1 && (k \geq 1) \\ R(L_{2k}) &= 2k - 3 && (k > 1) . \end{aligned}$$

Since

$$2F_k = F_{k+1} + F_{k-2} , \qquad 3F_k = F_{k+2} + F_{k-2} ,$$

we get

$$(5.9) \qquad R(2F_k) = 2 + 2[\tfrac{1}{2}(k-4)] \qquad (k \geq 4) ,$$

$$(5.10) \qquad R(3F_k) = 2 + 3[\tfrac{1}{2}(k-4)] \qquad (k \geq 4) .$$

The identity

$$L_{2j} F_k = F_{k+2j} + F_{k-2j}$$

yields

$$(5.11) \qquad R(L_{2j} F_k) = 2j + (2j+1)([\tfrac{1}{2}k] - j - 1) \qquad (k \geq 2j + 2) ;$$

for $j = 1$, (5.11) reduces to (5.10).

A few applications of (5.4) may be noted. For $k = 2$ we have

$$(5.12) \qquad R(F_1 + F_j + 2) = [\tfrac{1}{2}(i - j + 1)] + [\tfrac{1}{2}(j-3)][\tfrac{1}{2}(i - j + 2)] \quad (i - j \geq 2, j \geq 4).$$

while for $k = 3$ we have

(5.13) $$R(F_i + F_j + 2) = [\tfrac{1}{2}(i-j+1)] + [\tfrac{1}{2}(j-4)][\tfrac{1}{2}(i-j+2)]\ (i \geq j \geq 2,\ j \geq 5)\ .$$

Again, since

$$4F_k = F_{k+2} + F_k + F_{k-2}\ ,$$

it follows that

(5.14) $$R(4F_k) = 1 + 3[\tfrac{1}{2}(k-4)] \qquad (k \geq 4)\ .$$

Section 6

As remarked above, direct application of Theorem 1 leads to very complicated results for $R(N)$. If, however, all the k_s in the canonical representation of N have the same parity simpler results can be obtained. If all the k_s are odd then by (3.9),

(6.1) $$R(F_{k_1} + \cdots + F_{k_r}) = R(F_{k_1-1} + \cdots + F_{k_r-1})\ ;$$

we may therefore assume that all the k_s are even.

It will be convenient to introduce the following notation. Put

(6.2) $$N = F_{2k_1} + \cdots + F_{2k_r}\ ,$$

where

(6.3) $$k_1 > k_2 > \cdots > k_r \geq 1\ ;$$

also put

(6.4) $$j_s = k_s - k_{s-1} \qquad (s = 1, \cdots, r-1)\ ; \qquad j_r = k_r$$

and

$$(6.5) \qquad f_r = f(j_1, \cdots, j_r) = R(N) ,$$

where N is defined by (6.2).

Now by (3.20) and (3.9)

$$R(N) = R(F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) + (k_r - 1)R(F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}) .$$

By (6.4) and (6.5) this reduces to

$$(6.6) \qquad f(j_1, \cdots, j_r) = f(j_1, \cdots, j_{r-1}) + (j_r - 1)f(j_1, \cdots, j_{r-2}, j_{r-1} + 2) .$$

By (3.19) we have

$$\begin{aligned} &R(F_{2k_1-2k_r+2} + \cdots + F_{2k_{r-1}-2k_r+2}) \\ &\quad = R(F_{2k_1-2k_r} + \cdots + F_{2k_{r-1}-2k_r}) + R(F_{2k_1-2k_{r-1}+2} + \cdots + F_{2k_{r-2}-2k_{r-1}+2}) , \end{aligned}$$

so that

$$\begin{aligned} f(j_1, \cdots, j_{r-2}, j_{r-1} + 2) &= f(j_1, \cdots, j_{r-1}) + f(j_1, \cdots, j_{r-3}, j_{r-2} + 2) \\ &= f(j_1, \cdots, j_{r-1}) + f(j_1, \cdots, j_{r-2}) + \cdots + f(j_1) + 1 . \end{aligned}$$

Thus (6.6) reduces to

$$(6.7) \qquad f_r = f_{r-1} + (j_r - 1)(f_{r-2} + \cdots + f_1 + 1) .$$

If we define

$$(6.8) \qquad S_r = f_r + f_{r-1} + \cdots + f_1 + 1, \quad S_0 = 1 ,$$

then (6.7) becomes

$$f_r - f_{r-1} = (j_r - 1)S_{r-2} \qquad (r \geq 2)$$

and therefore

$$S_r - (j_r + 1)\, S_{r-1} + S_{r-2} = 0 \quad (r \geq 2)\,. \tag{6.9}$$

We may now state

Theorem 4. With the notation (6.2), (6.3), (6.4), (6.5), $f_r = R(N)$ is determined by means of (6.9) with $S_0 = 1$, $S_1 = j_1 + 1$ and

$$f_r = S_r - S_{r-1}\ .$$

The first few values of S_r are given by

$$S_0 = 1,\quad S_1 = j_1 + 1,\quad S_2 = j_1 j_2 + j_1 + j_2,\quad S_3 = j_1 j_2 j_3 + j_1 j_2 + j_1 j_3 + j_2 j_3 + j_2 - 1.$$

It is evident that $S_r = S(j_1, \cdots, j_r)$ is a polynomial in $j_1, \cdots, j_r$; indeed it is a continuant [1, vol. 2, p. 494].

We have for example

$$S_r = \begin{vmatrix} j_1 + 1 & -1 & 0 & \cdots & 0 \\ -1 & j_2 + 1 & -1 & \cdots & 0 \\ 0 & -1 & j_3 + 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & j_r + 1 \end{vmatrix}$$

and

$$S_r(j_1, j_2, \cdots, j_r) = S(j_r, j_{r-1}, \cdots, j_1)\ .$$

The latter formula implies

$$R(F_{2k_1} + \cdots + F_{2k_r}) = R(F_{2k'_1} + \cdots + F_{2k'_r})\ , \tag{6.10}$$

where

$$k_1' = k_r, \quad k_2' = k_1 - k_r, \quad k_3' = k_1 - k_{r-1}, \cdots, k_r' = k_1 - k_2 .$$

When

$$j_1 = j_2 = \cdots = j_r = j \quad , \tag{6.11}$$

we can obtain a simple explicit formula for S_r. Since in this case

$$S_r - (j+1)S_{r-1} + S_{r-2} = 0 \quad (r \geq 2) , \quad S_0 = 1, \quad S_1 = j+1 \quad ,$$

we find that

$$\sum_{r=0}^{\infty} S_r x^r = (1 - (j+1)x + x^2)^{-1} = \sum_{s=0}^{\infty} x^s (j+1-x)^s$$

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^t \binom{s}{t} (j+1)^{s-t} x^{s+t} \quad ,$$

which gives

$$\cdots \qquad S_r = \sum_{2t \leq r}^{\infty} (-1)^t \binom{r-t}{t} (j+1)^{r-2t} \quad . \tag{6.12}$$

In particular, for $j = 1$, (6.12) reduces to

$$S_r = r + 1 \qquad (j = 1) \quad . \tag{6.13}$$

For certain applications it is of interest to take

$$j_1 = \cdots = j_{r-1} = j, \qquad j_r = k \quad . \tag{6.14}$$

Then $S_1, \cdots, S_{r-1}$ are given by (6.12) while

$$S'_r = (k + 1) S_{r-1} - S_{r-2} \quad , \tag{6.15}$$

where $S'_r = S(j, \cdots, j, k)$. It follows from (6.15) that

$$f'_r = f(j, \cdots, j, k) = k S_{r-1} - S_{r-2} \, . \tag{6.16}$$

In view of the identity

$$L_{2j+1} F_{2k} = F_{2k+2j} + F_{2k+2j-2} + \cdots + F_{2k-2j}$$

we get, using (6.13) and (6.16),

$$R(L_{2j+1} F_{2k}) = (k - j)(2j + 1) - 2j \quad (k > j) \, . \tag{6.17}$$

For $k = j$ we have

$$R(L_{2j+1} F_{2j}) = 1 \quad . \tag{6.18}$$

Note that

$$L_{2j+1} F_{2j} = F_{4j+1} - 1, \quad L_{2j-1} F_{2j} = F_{4j-1} - 1 \, .$$

When $j = 2$, we have

$$\sum_{r=0}^{\infty} S_r x^{r+1} = \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n \quad ,$$

so that

$$S_r = F_{2r+2} \quad . \tag{6.19}$$

We now recall the identities

$$F_4 + F_8 + \cdots + F_{4n} = F_{2n+1}^2 - 1 \qquad (n \geq 1)\,,$$

$$F_2 + F_6 + \cdots + F_{4n-2} = F_{2n}^2 \qquad (n \geq 1)\,,$$

$$F_3 + F_7 + \cdots + F_{4n-1} = F_{2n}F_{2n+1} \qquad (n \geq 1)\,,$$

$$F_1 + F_5 + \cdots + F_{4n-3} = F_{2n}F_{2n-1} \qquad (n \geq 1)\,.$$

It follows readily, using (6.16) and (6.19) that

(6.20) $$R(F_{2n+1}^2 - 1) = F_{2n+1} \qquad (n \geq 0)\,,$$

(6.21) $$R(F_{2n}^2) = F_{2n-1} \qquad (n \geq 1)\,,$$

(6.22) $$R(F_{2n}F_{2n+1}) = F_{2n-1} \qquad (n \geq 1)\,,$$

(6.23) $$R(F_{2n}F_{2n-1}) = F_{2n-1} \qquad (n \geq 1)\,.$$

(6.24) $$R(F_{2n+1}^2 - 2) = F_{2n} \qquad (n \geq 1)\,,$$

(6.25) $$R(F_{2n}^2 - 1) = F_{2n} \qquad (n \geq 1)\,,$$

(6.26) $$R(F_{2n}F_{2n+1} - 1) = F_{2n} \qquad (n \geq 1)\,,$$

(6.27) $$R(F_{2n}F_{2n-1} - 1) = F_{2n-1}\,.$$

$\cdots$

Combining (6.20) with (6.24), and so on, we get

(6.28) $$R'(F_{2n-1}^2 - 1) = F_{2n} \qquad (n \geq 1)\,,$$

(6.29) $$R'(F_{2n}^2) = F_{2n+1} \qquad (n \geq 0)\,,$$

(6.30) $$R'(F_{2n}F_{2n+1}) = F_{2n+1} \qquad (n \geq 0)\,,$$

(6.31) $$R'(F_{2n}F_{2n-1}) = 2F_{2n-1} \qquad (n \geq 1)\,.$$

We have also

$$(6.32) \qquad R(F_{2n}^2 - 2) = F_{2n-2} \qquad (n \geq 1),$$

$$(6.33) \qquad R(F_{2n+1}^2) = F_{2n-1} \qquad (n \geq 1),$$

so that

$$(6.34) \qquad R'(F_{2n}^2 - 1) = L_{2n-1} \qquad (n \geq 1),$$

$$(6.35) \qquad R'(F_{2n+1}^2) = L_{2n} \qquad (n \geq 0).$$

Several of these results were obtained in [4].

In a similar way one can also prove the following formulas.

$$(6.36) \qquad R(F_{2n}F_{2m}) = R(F_{2n+1}F_{2m}) = (n - m)F_{2m} + F_{2m-1} \qquad (n \geq m),$$

$$(6.37) \qquad R(F_{2n}F_{2m+1}) = R(F_{2n+1}F_{2m+1}) = (n - m)F_{2m+1} \qquad (n > m).$$

Section 7

We shall now prove

Theorem 5. Let N have the canonical representation

$$(7.1) \qquad N = F_{k_1} + \cdots + F_{k_r}.$$

Then $e(N + 1) = e(N)$ if and only if $k_r = 2$.

Proof. Take $k_r = 2$. Then

$$N + 1 = F_{k_1} + \cdots + F_{k_{r-1}} + F_3,$$

so that

$$e(N + 1) = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + F_2.$$

Since

$$e(N) = F_{k_1-1} + \cdots + F_{k_{r-1}-1} + F_1 ,$$

it follows that $e(N+1) = e(N)$.

Now take $k_r > 2$. Then

$$N + 1 = F_{k_1} + \cdots + F_{k_r} + F_2$$

and

$$e(N+1) = F_{k_1-1} + \cdots + F_{k_r-1} + 1 .$$

But

$$e(N) = F_{k_1-1} + \cdots + F_{k_r-1} < e(N+1) .$$

This completes the proof of the theorem.

If N is defined by (7.1) then

$$M = F_{k_1+1} + \cdots + F_{k_r+1}$$

satisfies $e(M) = N$. Moreover, by the last theorem, if $k_r = 2$ then also $e(M-1) = N$.

Consider

$$N + 1 = F_{k_1+1} + \cdots + F_{k_r+1} + F_2 .$$

Clearly

$$e(M+1) = F_{k_1} + \cdots + F_{k_r} + 1 = N + 1 .$$

Also, since $F_3 = 2$, we have

$$M - 2 = F_{k_1+1} + \cdots + F_{k_{r-1}} + 1 ,$$

$$e(M-2) = F_{k_1} + \cdots + F_{k_{r-1}} = N - 1 .$$

It follows that one can have at most two consecutive numbers N, $N + 1$, such that $e(N) = e(N + 1)$. This justifies the assertion about $A(m,n)$ in the introduction.

Section 8

Put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2} .$$

Then it is easily verified that

$$\alpha^{-1}F_n = F_{n-1} - \beta^n . \tag{8.1}$$

Hence if N has the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} ,$$

it follows that

$$e(N) - \alpha^{-1} N = \beta^{k_1} + \beta^{k_2} + \cdots + \beta^{k_r} . \tag{8.2}$$

Consequently

$$\begin{aligned} \left| e(N) - \alpha^{-1}N \right| &\le \alpha^{-k_1} + \alpha^{-k_2} + \cdots + \alpha^{-k_r} \\ &\le \alpha^{-2} + \alpha^{-4} + \cdots + \alpha^{-2r} \\ &< \frac{\alpha^{-2}}{1 - \alpha^{-2}} + \frac{1}{\alpha^2 - 1} = \frac{1}{\alpha} < 0.62 . \end{aligned}$$

If we put

$$\alpha^{-1}N = [\alpha^{-1}N] + \in \qquad (0 < \in < 1) ,$$

where $[\alpha^{-1}N]$ denotes the greatest integer $\le \alpha^{-1}N$, then

$$-0.62 < e(N) - [\alpha^{-1}N] - \in < 0.62 .$$

This implies

$$[\alpha^{-1}N] \le e(N) \le [\alpha^{-1}N] + 1 . \tag{8.3}$$

If $k_r \ge 3$ it follows from (8.2) that

$$|e(N) - \alpha^{-1}N| \le \alpha^{-3} + \alpha^{-5} + \cdots + \alpha^{-2r-1}$$

$$< \frac{\alpha^{-3}}{1 - \alpha^{-2}} = \frac{1}{\alpha(\alpha^2 - 1)} = \frac{1}{\alpha^2} < \frac{1}{2}$$

and therefore

$$e(N) = \{\alpha^{-1}N\} \qquad (k_r > 2) , \tag{8.4}$$

where $\{\alpha^{-1}N\}$ denotes the integer nearest to $\alpha^{-1}N$.

Thus the value of $e(N)$ is determined by (8.4) except possibly when $k_r = 2$. Now when $k_r = 2$ we have as above

$$\cdots e(N) - \alpha^{-1}N \ge \alpha^{-2} - \alpha^{-5} - \alpha^{-7} - \cdots - \alpha^{-2r-1} > \alpha^{-2} - \frac{\alpha^{-5}}{1 - \alpha^{-2}}$$

$$= \frac{1}{\alpha^2} - \frac{1}{\alpha^3(\alpha^2 - 1)} = \frac{1}{\alpha^2} - \frac{1}{\alpha^4} = \frac{1}{\alpha^3} > 0 ,$$

so that

$$0 < e(N) - \alpha^{-1}N < 0.62 .$$

It therefore follows that

$$e(N) = [\alpha^{-1}N] + 1 \qquad (k_r = 2) . \tag{8.5}$$

We may now state

Theorem 6. Let N have the canonical representation

$$N = F_{k_1} + \cdots + F_{k_r} .$$

Then if $k_r > 2$,

$$e(N) = \{\alpha^{-1}N\} , \tag{8.6}$$

the integer nearest $\alpha^{-1}N$; if $k_r = 2$,

$$e(N) = [\alpha^{-1}N] + 1 . \tag{8.7}$$

We remark that (8.6) and (8.7) overlap. For example for

$$N = 6 = F_5 + F_2, \quad e(6) = F_4 + F_1 = 4, \quad [6\alpha^{-1}] = [3.72] = 3,$$

$$\{6\alpha^{-1}\} = \{3.72\} = 4.$$

However for

$$N = 25 = F_8 + F_4 + F_2, \quad e(25) = F_7 + F_3 + F_1 = 16, \quad [25\alpha^{-1}] = 15,$$

$$\{25\alpha^{-1}\} = 15 .$$

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