Local Construction of Planar Spanners in Unit Disk Graphs with Irregular Transmission Ranges

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March 30, 2005

Abstract

We give an algorithm for constructing a spanner of a wireless network modeled as a unit disk graph with nodes of irregular transmission ranges, whereby for some parameter $0 < r \leq 1$ the transmission range of a node includes the entire disk around the node of radius at least r and it does not include any node at distance more than one. The construction of a spanner is distributed and local in the sense that

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**Research supported in part by NSERC (Natural Science and Engineering Research Council of Canada) grant.

[††]Research supported in part by MITACS (Mathematics of Information Technology and Complex Systems) grant.

nodes use only information at their vicinity, moreover for a given integer $k \geq 2$ each node needs only consider all the nodes at distance at most k hops from it. The resulting spanner has maximum degree at most $3 + \frac{6}{\pi r} + \frac{r+1}{r^2}$, when $0 < r < 1$ (and at most five, when $r = 1$). Furthermore it is shown that the spanner is planar provided that the distance between any two nodes is at least $\sqrt{1 - r^2}$. If the spanner is planar then for $k \geq 2$ the sum of the Euclidean lengths of the edges of the spanner is at most $\frac{kr+1}{kr-1}$ times the sum of the Euclidean lengths of the edges of a minimum weight Euclidean spanning tree.

1 Introduction

The problem of constructing spanners (e.g., minimum cost spanning trees, triangulated spanners, planar spanners) for "various types" of geometric graphs has been considered extensively in the current literature due to its many applications ranging from VLSI design, to efficient communication in networks and medical imaging (see Eppstein [8]). A variety of optimization results have been derived that considered tradeoffs among weight, diameter, dilation, and max degree between the original graph and the resulting spanner. Nevertheless the majority of these results (e.g., Eppstein [8], Arya, Das, Mount, Salowe, and Smid [1], Arya and Smid [2], Narasimhan and Smid [18], Bose, Gudmundsson and Smid [4]) consider only centralized, non-distributed algorithms that do not take into account the dynamic changes taking place in a communication network.

In recent years, the problem of producing efficiently a planar spanner has been given new research impetus in communication networks due to its applicability in more dynamically changing environments consisting of wireless interconnected nodes. In this case, in addition to considering the previously mentioned parameters of weight, diameter, dilation, and max degree, a new condition of *locality of communication* becomes important: nodes should take into account information by consulting only other nodes within their "close" geographic vicinity. In fact locality in wireless networking is a necessity imposed by the geographic limitations of the networking environment.

Moreover, there are two important issues in wireless networking. The first one is to be able to perform locally and efficiently important communication tasks, like routing. Ultimately, this is easily resolved if the underlying graph is planar using face routing (see Kranakis, Singh, and Urrutia [11]). The second one is a "local" construction of a "simple" planar spanner from the given wireless network. In fact, Bose, Morin, Stojmenovic, and Urrutia [5] address this problem for wireless networks corresponding to unit disk graphs by constructing a planar spanner in a local and distributed manner using the Gabriel test (see Gabriel and Sokal [9]).

In addition to the Gabriel test, there are known algorithms for constructing locally and distributively a planar subgraph of bounded degree and constant stretch factor for unit disk graphs. However the resulting degree is rather high (more than 25), the constructions are relatively complicated, and the cost of such graph can be much higher than the cost of a Minimum cost Spanning Tree (MST) (e.g., see Li, Calinescu, and Wan [15], Wang and Li [20], Li and Wang [16]). In a recent paper Li, Wang, and Song [17] give an algorithm for constructing a spanner from the relative neighborhood graph [19] of a unit disk graph. This spanner has maximum degree at most six, and its total weight is a constant multiple of the total weight of the MST, where the weight of a graph is defined as the sum of Euclidean lengths of the edges. In this paper we consider the problem of constructing a spanner of networks which are more general than those represented by a unit disk graph.

A unit disk graph is a representation of a wireless network in which all nodes have the same circular transmission range. Clearly, this is an idealized representation and it does not need to correspond to actual situations. Typically, the nodes in a network are not exactly identical and some obstacles in the terrain containing the nodes may result in the transmission ranges of nodes to be irregular. In this paper we are taking into consideration the fact that the transmission range of each node of a network could be irregular to "some degree" (see Figure 1). We assume that in a given network there is an additional parameter r, a positive real number less than or equal 1. The transmission range of a node in the network is assumed to be a region contained within the unit disk around the node, but this region contains all points at distance less than r. Thus any two nodes at distance at most r can communicate directly, but no nodes at distance more than 1 can communicate directly. Two nodes at distance more than r and at most 1 may or may not communicate directly. An example of a transmission range of node u is shown in Figure 1 as the darker area. We shall consider the static case in which the irregularity of each node is fixed and does not change with time. We call the geometric representation of such a network a *unit disk graph with irregularity* r. This class of unit disk graphs with irregular transmission ranges was first introduced by Barrière, Fraigniaud, Narayanan, and

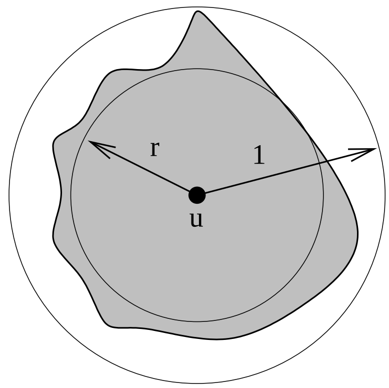

Figure 1: The irregular transmission area of node u.

Opatrny [3] in order to propose robust position-based routing. The problem of constructing a spanner for unit disk graphs with irregular transmission ranges is more complex, for example the usual planarization algorithms like the Gabriel test or the relative neighborhood graph algorithm do not work for them.

1.1 Results and outline of the paper

We give an algorithm for constructing a spanner of a connected unit disk graph with irregularity r. The construction is *local* in the sense that nodes use only information at their vicinity: given $k \geq 2$, each node needs only to consider all the nodes at distance at most k hops from it, i.e., nodes connected to it by paths of length at most k. The resulting spanner has maximum degree at most $3 + \frac{6}{\pi r} + \frac{r+1}{r^2}$, when $0 < r < 1$ (and at most five, when $r = 1$). Moreover, it is shown that the spanner is *planar* provided that the distance between any two nodes is at least $\sqrt{1 - r^2}$. For $k \geq 2$ the sum of the euclidean lengths of the edges of the spanner is at most $\frac{kr+1}{kr-1}$ times the sum of the euclidean lengths of the edges of a minimum weight euclidean spanning tree if the spanner is planar. The class of graphs whereby the distance between any two nodes is at least λ (in our graphs $\lambda = \sqrt{1 - r^2}$) were first called *civilized* by Doyle and Snell [7][page 136] and have also been referred to as λ-*precision* by Hunt, Marathe, Radhakrishnan, Ravi, Rosenkrantz, and Stearns [10], and $\Omega(1)$-*constant* by Kuhn, Wattenhofer, and Zollinger [14] (see also Kuhn, Wattenhofer, Zhang, and Zollinger [13]).

Our results extend work of Li, Wang, and Song [17] mentioned above from the case $r = 1$ to arbitrary irregularity factor r. Note that even in the special case $r = 1$, we obtain explicit bounds on degree and cost of the spanner rather that asymptotic bounds. Also, our proofs use only elementary techniques and do not rely on [6].

An outline of the paper is as follows. Section 2 gives definitions needed for the algorithm, the main one being a definition of a linear order of the edges of the graph, while Section 3 gives the main result on constructing a spanner and proves the correctness of our algorithm. In Section 4 we provide several examples and compare our result to other constructions of spanners in the literature.

2 Preliminaries

A graph G is *geometric* if it is embedded into the Euclidean plane and the edges are straight line segments between the nodes. The edge selection in our algorithms will depend on a linear order on edges of the input geometric graph G.

2.1 Linear Order on Edges

Let $|u, v|$ denote the Euclidean distance between nodes u and v. Intuitively, we can define a linear order on the edges of G

- by first considering the Euclidean length,
- if two edges have the same length, the one with rightmost, topmost node is larger, and finally
- if two edges of same length share their rightmost, topmost node, then their second endnode is considered; the edge with the right most, top most second endnode is defined as larger.

Formally, we have the following definition.

Definition 1 (**Compatible Linear Order.**) *Each edge $\{u, v\}$ is assigned a 5-tuple $(|u, v|, x_1, y_1, x_2, y_2)$, where x_1, y_1 and x_2, y_2 are the coordinates of the end-points of the edge with either $x_1 > x_2$ or $x_1 = x_2$ and $y_1 > y_2$. Clearly this gives a unique 5-tuple to any edge, and 5-tuples assigned to any two* *edges are distinct. The linear order $\prec$ is defined by using the lexicographical ordering of the assigned 5-tuples.*

Notice that in the order $\prec$, we first consider the Euclidean length of edges and the coordinates are used for ordering edges of the same length. The input graph G may have many minimum cost spanning trees (MSTs) when the Euclidean length of edges is the cost function. However, if we break the ties by the linear order $\prec$, then G has a unique MST $T^{\prec}$ which can be computed for example by Kruskal's algorithm.

Definition 2 *For a given geometric graph H, define cost(H) as the sum of Euclidean lengths of the edges of H.*

Definition 3 *Given a graph G and a vertex v of G, we denote by $N_k[v]$ the distance k closed neighborhood of v, i.e. the nodes of G reachable from v by a path with at most k edges. Note that $v \in N_k[v]$. Sometimes, the graph induced by vertices in $N_k[v]$ will be denoted by the same symbol $N_k[v]$.*

3 Constructing a Spanner

This section is the core of our paper. Subsection 3.1 gives the main algorithm for constructing spanners directly from a unit disk graph, while Subsection 3.2 states the main theorem (Theorem 1) and its complete proof.

3.1 Spanner algorithm

Consider algorithm **LocalMST**$_k$, for $k \geq 2$.

Algorithm: LocalMST$_k$
Input: A connected geometric graph G with the linear order $\prec$;
Output: Graph $G_k^{\prec}$

Run the following algorithm at each node v of G:

1. Learn your distance k neighborhood $N_k[v]$.
2. Construct locally the unique MST $T_k(v)$ of $N_k[v]$.
3. Broadcast in $N_1[v]$ the edges of $N_[1]v$ which have been retained in $T_k(v)$ (i.e. $N_1[v] \cap T_k(v)$).
4. The output graph $G_k^{\prec}$ is defined as follows: an edge is selected into $G_k^{\prec}$ if and only if it was retained by both of its incident nodes.

Clearly, this is a distributed algorithm. To learn its distance k neighborhood, v first broadcasts its coordinates to all its neighbors. After having learnt its distance k neighborhood it broadcasts it to all its neighbours. It can then construct the unique MST $T_k(v)$ (which is selected using Kruskal's algorithm [12] and the linear order $\prec$) of $N_k[v]$ and broadcasts edges in $N_1[v] \cap T(v)$ to all nodes in $N_1[v]$. The parameter k determines the desired locality of our algorithm, and thus the resulting graph $G_k^{\prec}$ is constructed "locally", each node v uses only knowledge of $N_k[v]$ and the results of its neighbours.

3.2 Main result and proof of correctness

Let G be a unit geometric graph with irregularity r and $k \geq 2$. We show that the graph $G_k^{\prec}$ constructed by the above algorithm has interesting properties summarized in the following theorem.

Theorem 1 *If G is a connected unit geometric graph with irregularity r and $k \geq 2$, then the graph $G_k^{\prec}$ has the following properties.*

a) $G_k^{\prec}$ is connected;

b) if the distance between any two nodes of the network is at least $\sqrt{1-r^2}$, then the graph $G_k^{\prec}$ is planar;

c) $\Delta(G_k^{\prec}) \le \begin{cases} 5 & \text{if } r = 1, \\ 3 + \frac{6}{\pi r} + \frac{r+1}{r^2} & \text{if } 0 < r < 1; \end{cases}$

d) If $G_k^{\prec}$ *is planar and* $kr > 1$*, then* $cost(G_k^{\prec}) \le \frac{kr+1}{kr-1} \times cost(T^{\prec})$*.*

Proof. The proof of part a) follows from the following claim.

Claim 1 $T^{\prec} \subseteq G_k^{\prec}$.

Proof. We argue by contradiction. Let the edge $\{u, v\}$ be retained in T, but rejected in $G_k^{\prec}$. Without loss of generality we may assume it was rejected in $T_k(v)$. Since $\{u, v\}$ was retained in T, there is no other path in T joining u and v. Since $\{u, v\}$ was rejected by $T_k(v)$, there exists a path, say p, in $T_k(v)$ joining u and v and using only edges smaller than $\{u, v\}$. Let $\{w, w'\}$

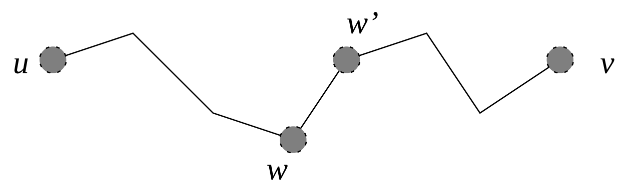

Figure 2: Path p from u to v.

be an edge in p such that $\{w, w'\} \notin T$ (see Figure 3.2). It follows that there is a path in T joining w and w' and using only edges smaller than the edge $\{w, w'\}$. As this argument applies to each such edge of p, there must be a path in T joining u and v using only edges smaller then $\{u, v\}$. This contradicts the fact that the edge $\{u, v\}$ was retained in T. □

To prove part b), assume by way of contradiction, that $G_k^{\prec}$ is not planar and let $\{u, v\}$ and $\{w, t\}$ be two crossing edges in $G_k^{\prec}$. Without loss of generality we may assume that the angle $\angle uwv$ is the largest angle in the quadrilateral $uwut$ (see Figure 3). Clearly, this angle is at least $\pi/2$. Since $|u, v| \le 1$ we have $|u, w|^2 + |w, v|^2 \le 1$. Thus $|u, w|^2 \le 1 - |w, v|^2 \le r^2$ since $|w, v| \ge \sqrt{1 - r^2}$ by our assumption. Therefore, $\{u, w\}$ is an edge in G. The same argument shows that $\{w, v\}$ is an edge in G.

We will show that the diagonal $\{u, v\}$ will not be selected into $G_k^{\prec}$ by u. Assume u computes $T_k(u)$ using Kruskal's algorithm. Either $\{u, w\}$ is retained in $T_k(u)$, or there already exists a path in $T_k(u)$ consisting of smaller

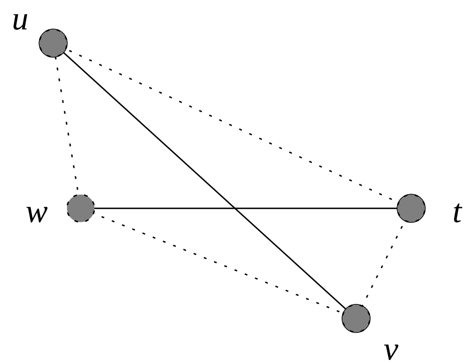

Figure 3: Two crossing edges in $G_k^{\prec}$.

edges connecting u and w. Analogously, the same is true for $\{w, v\}$. This means that at the moment $\{u, v\}$ is considered by u for inclusion into $T_k(u)$, there already exists a path in $T_k(u)$ connecting u and v and hence $\{u, v\}$ will be rejected by u, which contradicts the fact that edge $\{u, v\}$ is in $G_k^{\prec}$. Note that from our assumption on distance between vertices of G and the assumption that G is connected, we have $\sqrt{1 - r^2} \leq r$ and thus $r \geq \sqrt{\frac{1}{2}}$.

To prove part c), let u be any vertex of G. Partition the unit circle around u into six equal size sectors each with angle at u equal to $\pi/3$. Figure 4 depicts such a sector by the dark shaded area. Since G is finite, we may assume that the edges of these sectors do not pass through any neighbor of u. Hence, for any two neighbors v and w of u inside any fixed sector, the angle $\angle wuv$ is less that $\pi/3$. Then $|v, w| < \max\{|u, v|, |u, w|\}$. If $|v, w| \leq r$, then one of $\{u, v\}$, $\{u, w\}$ would have been replaced in $G_k^{\prec}$ by $\{v, w\}$. Thus, we conclude $|v, w| > r$. If $r = 1$, it follows that u can have at most one neighbor inside of each sector. So u has at most six neighbors in $G_k^{\prec}$. Suppose u has six neighbors. However, this may only occur if u is in the center of a perfect hexagon formed by its neighbors. However, in this case only two incident edges will be retained, as four of the incident edges will be deleted as the largest edges of an incident equilateral triangle. Hence u has at most five neighbors as claimed.

Suppose now, $r < 1$. Consider a fixed sector S defined above. Draw a circle of radius $r/2$ around every neighbor of u in this sector. It follows that these circles are disjoint and all are inside the region determined by the union of the circle of radius $r/2$ centred at u, the sector of radius $1 + r/2$ centred at u and containing the sector S, and two rectangles with sides $1 + r/2$ and $r/2$ (see Figure 4). Hence an upper bound on the number of neighbors of u

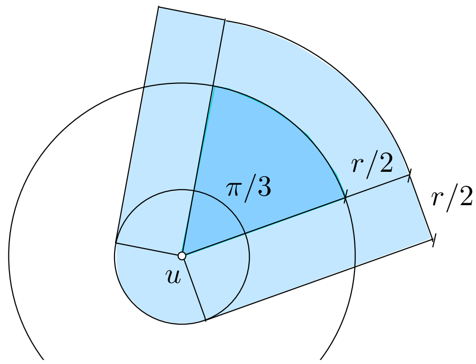

Figure 4: The light shaded area contains all disjoint circles of radius $r/2$ around all neighbors of u inside the dark shaded area.

in the sector S is the number of circles of radius $r/2$ that can be packed into this area. This number is at most

$$\frac{\pi(r/2)^2 + \frac{\pi(1+r/2)^2}{6} + 2(1+r/2)r/2}{\pi r^2} < \frac{1}{2} + \frac{1}{\pi r} + \frac{r+1}{6r^2}.$$

Summing up through all six sectors, we obtain that u has at most $3 + \frac{6}{\pi r} + \frac{r+1}{r^2}$ neighbors.

The following claim captures a crucial property of the graph $G_k^{\prec}$ that helps to prove part d).

Claim 2 *Every cycle C in $G_k^{\prec}$ has the Euclidean length greater than* $\max\{(k+1)r, kr + l\}$*, where l is the length of a longest edge in C.*

Proof. Let C be a cycle in $G_k^{\prec}$ and l be the Euclidean length of the largest edge, say $\{u, v\}$, of this cycle. Without loss of generality we may assume that v is counterclockwise from u (see Figure 5). Since C was retained in $G_k^{\prec}$, the edge $\{u, v\}$ must have been retained in $T_k(u)$. However, this means that there exists a node $z \in C$ such that $z \notin N_k[u]$, otherwise u would have seen the whole cycle C and therefore rejected edge $\{u, v\}$ as the largest edge of C. Since $z \notin N_k[u]$, the path from u to z clockwise around C contains a vertex

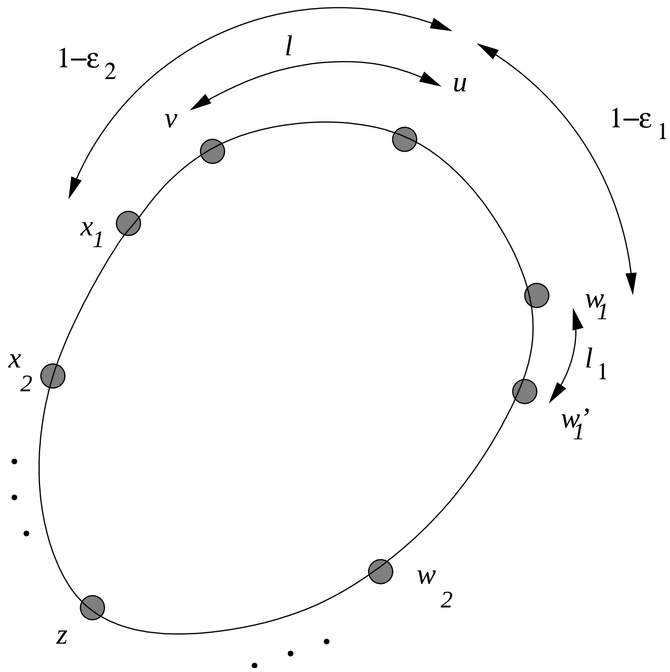

Figure 5: Cycle C.

of $N_i[u] \setminus N_{i-1}[u]$ for all $1 \leq i \leq k$. Let w_i be the furthest such a vertex in $N_i[u] \setminus N_{i-1}[u]$. Similarly, also the path from u to z counter clockwise around C contains a vertex of $N_i[u] \setminus N_{i-1}[u]$ for all $1 \leq i \leq k$, and let x_i be the furthest such a vertex in $N_i[u] \setminus N_{i-1}[u]$. It follows that $w_1, w_2, \ldots, w_k$ are in clockwise order around C while $x_1, x_2, \ldots, x_k$ are is counter clockwise order.

By definition, if $k \geq 2$ then the Euclidean distances $|u, w_2|$, $|u, x_2|$, $|z, w_{k-1}|$, $|z, x_{k-1}|$, and for all $1 \leq i \leq k-2$ $|w_i, w_{i+2}|$ and $|x_i, x_{i+2}|$ are all greater than r. From the triangle inequality, we have.

If k is odd then the Euclidean length

$$\begin{aligned} |C| &\geq |u, w_{k-1}| + |w_{k-1}, z| + |u, x_{k-1}| + |x_{k-1}, z| \\ &> \frac{k-1}{2} r + r + \frac{k-1}{2} r + r \\ &= (k+1) r, \end{aligned}$$

or similarly

$$|C| \geq |u, w_2| + |w_2, w_{k-1}| + |w_{k-1}, z| + |u, x_{k-1}| + |x_{k-1}, z|$$

$$
\begin{aligned}
&> \; l + \frac{k-3}{2} r + r + \frac{k-1}{2} r + r \\
&= \; kr + l,
\end{aligned}
$$

If k is even then the Euclidean length

$$
\begin{aligned}
|C| \; &\geq \; |w_1, x_1|_C + |w_1, w_{k-1}| + |w_{k-1}, z| + |x_1, x_{k-1}| + |x_{k-1}, z| \\
&> \; |w_1, x_1| + \frac{k-2}{2} r + r + \frac{k-2}{2} r + r + r \\
&= \; |w_1, x_1|_C + kr \\
&\geq \; kr + l,
\end{aligned}
$$

where $|w_1, x_1|_C$ denotes the Euclidean length of the counterclockwise path from w_1 to x_1 on the cycle C. To complete the proof it remains to show that $|w_1, x_1|_C > r$. Let w_1' be the clockwise successor of w_1 along C. By definition of w_1 and the fact that $z \notin N_k[u]$, the vertex $w_1' \notin N_1[u]$. Let $|w_1, w_1'| = l_1$. We have $|w_1, u|_C \geq r - \epsilon_1$ and $|u, x_1|_C \geq r - \epsilon_2$ for some $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

From the triangle inequality and the fact that $w_1' \notin N_1[u]$, we get $r - \epsilon_1 + l_1 > r$. From this and the fact that l is the largest edge in C, we get $l \geq l_1 > \epsilon_1$. Since v is reachable from u, we know that $r - \epsilon_2 \geq l$. Combining with $l > \epsilon_1$ we get $\epsilon_1 + \epsilon_2 < r$ and thus $|w_1, x_1|_C \geq r - \epsilon_1 + r - \epsilon_2 > r$. $\square$

Finally we prove that if $G_k^{\prec}$ is planar, then $\text{cost}(G_k^{\prec}) \leq \frac{kr+1}{kr-1} \times \text{cost}(T^{\prec})$. Let $C_1, C_2, \ldots, C_f$ be the faces in $G_k^{\prec}$. First note that the sum of the Euclidean lengths of the faces is equal to twice the sum of the Euclidean lengths of all edges. This implies that $\text{cost}(G_k^{\prec})$ is equal to half the sum of the Euclidean lengths of the faces, which by Claim 2 is bounded from below by $(krf + \sum_{i=1}^{f} l_i)/2$ where l_i is the longest edge in C_i.

Since, $T^{\prec} \subseteq G_k^{\prec}$, it follows from the well-known Euler's formula that the spanning tree $T^{\prec}$ can be obtained by deleting some $f-1$ edges $e_1, e_2, \ldots, e_{f-1}$ from $G_k^{\prec}$. Therefore we obtain that $\text{cost}(G_k^{\prec}) \leq \text{cost}(T^{\prec}) + \sum_{j=1}^{f-1} |e_j|$. We want to upper bound the last sum by $\sum_{i=1}^{f} l_i$. To do this, we need to assign each edge e_j to a unique face C_i so that $e_j \in C_i$. For this, consider the bipartite graph H with partite sets $X = \{e_1, e_2, \ldots, e_{f-1}\}$ and $Y = \{C_1, C_2, \ldots, C_f\}$ in which a vertex in X is joined by an edge to the two faces it is incident on. Consider a subset $X' \subseteq X$. We claim that $|N(X')| > |X'|$. Indeed, if for some X', the edges in X' are incident only to $|X'|$ faces, then after removal of these edges we obtain a new planar graph which will have the same number of nodes, will have $|X'|$ less edges and

$|X'| - 1$ less faces that $G_k^{\prec}$, which is a contradiction with Euler's formula. It follows from the well-known Hall's matching theorem that H has a matching saturating X. Now, assign the edge e_j to the face determined by the matching. We may assume (after appropriate relabelling) that e_j is assigned to C_j for $j = 1, \ldots, f-1$. Since the length of a longest edge in C_j is l_j, we have $\sum_{j=1}^{f-1} |e_j| \leq \sum_{j=1}^{f-1} l_j$, and hence $\text{cost}(G_k^{\prec}) \leq \text{cost}(T^{\prec}) + \sum_{j=1}^{f-1} l_j$. This implies that

$$\begin{aligned}
\text{cost}(T^{\prec}) &\geq krf/2 - \sum_{i=1}^{f-1} l_i/2 \\
&\geq (krf - f + 1)/2).
\end{aligned}$$

Notice that by the assumption, $kr > 1$ and hence the last expression is positive. Consequently,

$$\begin{aligned}
\frac{\text{cost}(G_k^{\prec})}{\text{cost}(T^{\prec})} &\leq \frac{\text{cost}(T^{\prec}) + \sum_{j=1}^{f-1} l_j}{\text{cost}(T^{\prec})} \\
&\leq 1 + \frac{f-1}{\text{cost}(T^{\prec})} \\
&\leq 1 + \frac{f}{(krf - f + 1)/2} \\
&\leq 1 + \frac{2}{kr - 1} \\
&\leq \frac{kr + 1}{kr - 1}
\end{aligned}$$

This completes the proof of the theorem. □

To see that $G_k^{\prec}$ is not necessarily planar, consider the example of a graph G on Figure 6 for which any connected spanner must retain all edges.

Corollary 1 *If G is a connected unit disk graph and $k \geq 2$, then the graph $G_k^{\prec}$ has the following properties.*

a) $G_k^{\prec}$ is connected;

b) $G_k^{\prec}$ is planar;

c) $\Delta(G_k^{\prec}) \leq 5$;

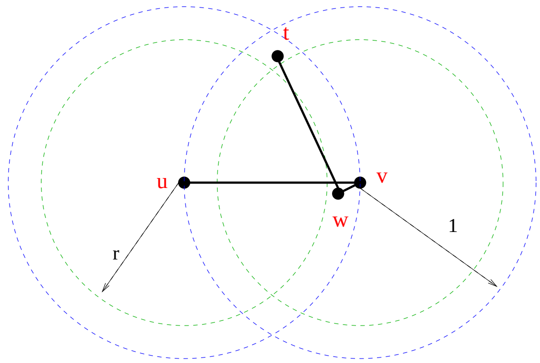

Figure 6: Configuration of vertices that gives a non-planar spanner.

d) $cost(G_k^{\prec}) \leq \frac{k+1}{k-1} \times cost(T^{\prec})$.

Thus, this corollary give identical results as those of Li, Wang, and Song [17] as far as the connectedness and the planarity is concerned, but it improves the maximum degree to 5 and gives an explicit value of the cost factor of the spanner.

4 Sharpness of Our Results

In this section, we look at the degree, cost, and connectivity of the resulting spanner on some examples of graphs, further illuminate the necessity of choosing a linear order on the edges, and compare our results with the RNG (Relative Neighborhood) [19] and GG (Gabriel) [9] graphs.

In RNG (respectively, GG), an edge between u and v exists if and only if the intersection of the circles centered at u and v contains no other point from the given pointset (respectively, the circle centered at $\frac{u+v}{2}$ and radius $\frac{|u,v|}{2}$ contains no other point from the given pointset). It is easy to see that if G is a unit disk graph with irregularity r then a local computation of both, RNG and GG can give a disconnected graph in some cases. We therefore make a comparison for the case when G is a unit disk graph with irregularity $r = 1$.

It is known that both the RNG and GG contain a minimum spanning tree and are contained in the Delaunay triangulation. Both RNG and GG have possibly unbounded degree and unbounded cost/cost(MST) as illustrated below.

4.1 Unbounded degree of GG and RNG

Consider a wheel W_n with n spokes (see W_8 in Figure 7). RNG and CG leave it unchanged, resulting in an unbounded degree of these spanners, while our algorithm leaves only one (the smallest) spoke.

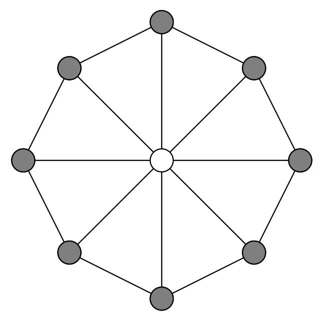

Figure 7: A wheel W_8 with 8 spokes.

4.2 Unbounded cost of GG and RNG

Concerning the unbounded cost, consider a $2 \times n$ mesh with vertical edges connecting points $(i/k, 0)$ and $(i/k, 1)$, for $i = 1, \ldots, n$, which are k times longer than horizontal ones. Both RNG and GG will retain all vertical edges. However, our algorithm will leave only the smallest (leftmost) vertical edge.

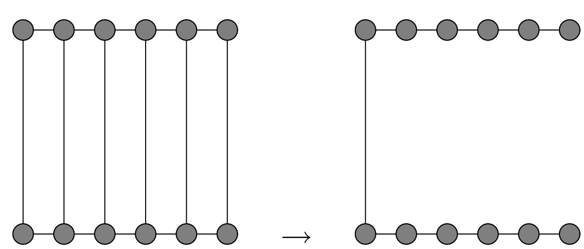

Figure 8: Applying our algorithm to a $2 \times n$ mesh.

4.3 Lower bound on the degree

We should also mention that five is indeed a lower bound on the degree of the resulting spanner. To see this, consider the star depicted in Figure 9 with a

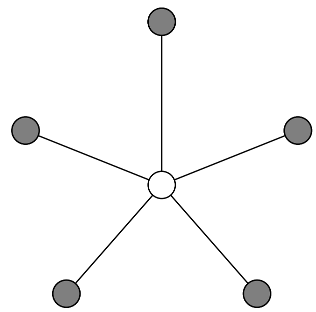

Figure 9: A star with 5 spokes.

central node and five equally distributed satellite nodes located at the nodes of a regular pentagon and at distance 1 from the center. If the nodes have reachability radius 1 then all five links of the central node must be retained in order to ensure the connectivity.

4.4 Necessity of using a linear order on edges

Observe that without the ordering on the edges of the geometric graph, the algorithm to obtain G does not work, because it could produce a disconnected graph. A simple counterexample with four nodes is depicted in Figure 10. It consists of four nodes v_1, v_2, v_3, v_4 such that the distance between any pair of them, but one (say v_1 and v_4) is equal to 1. Our nodes are the vertices of two equilateral triangles with disjoint interiors that share an edge (in this case $\{v_2, v_3\}$). Without a total ordering induced on the edges of this graph, we can get a disconnected graph.

5 Conclusions

In this paper, we gave a new local, distributed algorithm for constructing a planar spanner of a connected unit disk graph with nodes having irregular transmission ranges, give bounds on the degree of the spanner, and a sufficient

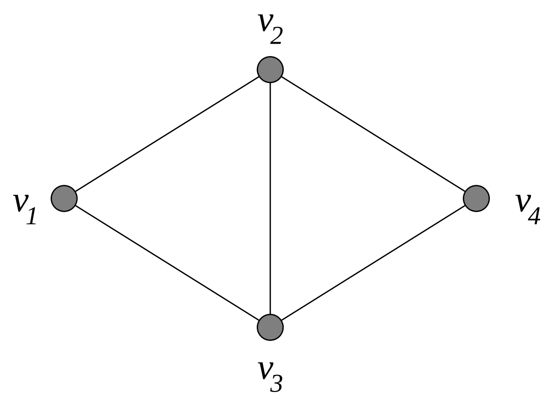

Figure 10: Two equilateral triangles.

condition on the graph to obtain a planar spanner. When the spanner is planar, we give an explicit bound on the the cost factor of the spanner. It would be interesting to derive a cost factor in case when the spanner is not planar. Another interesting problem is to see whether our techniques can be extended to obtain a distributed algorithm that constructs a low cost spanner of a given geometric unit graph (possibly with irregular transmission range) which in addition guarantee a low geometric stretch factor of edges.

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