Evacuating Robots from an Unknown Exit in a Disk

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Abstract. Consider k mobile robots inside a circular disk of unit radius. The robots are required to evacuate the disk through an unknown exit point situated on its boundary. We assume all robots having the same (unit) maximal speed and starting at the centre of the disk. The robots may communicate in order to inform themselves about the presence (and its position) or the absence of an exit. The goal is for all the robots to evacuate through the exit in minimum time.

We consider two models of communication between the robots: in *non-wireless* (or *local*) *communication* model robots exchange information only when simultaneously located at the same point, and *wireless communication* in which robots can communicate between each other at any time.

We study the following question for different values of k: what is the optimal evacuation time for k robots? We provide algorithms and show lower bounds in both communication models for $k = 2$ and $k = 3$ thus indicating a difference in evacuation time between the two models. We also obtain almost-tight bounds on the asymptotic relation between evacuation time and team size, for large k. We show that in the local communication model, a team of k robots can always evacuate in time $3 + \frac{2\pi}{k}$, whereas at least $3 + \frac{2\pi}{k} - O(k^{-2})$ time is sometimes required. In the wireless communication model, time $3 + \frac{\pi}{k} + O(k^{-4/3})$ always suffices to complete evacuation, and at least $3 + \frac{\pi}{k}$ is sometimes required. This shows a clear separation between the local and the wireless communication models.

Key Words and Phrases: Cycle, Disk, Evacuation, Mobile Robot, Speed.

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1 Introduction

Consider a team of mobile robots inside an environment represented by a circular disk of unit radius. The robots need to find an exit being a point at an unknown position on the boundary of the disk in order to evacuate through this point. The exit is recognized when visited by a robot. The robots may communicate in order to exchange the knowledge about the presence (or the absence) of the exit acquired through their previous movements. We consider two communication models. In the *non-wireless* (or *local*) model, communication is possible between robots which arrive at the same point (in the environment) at the same moment, while the *wireless* model allows broadcasting a message by a robot, which is instantly acquired by other robots, independently of their current positions in the environment. The robots start at the centre of the disk and they can move with a speed not exceeding their maximum velocity (which is the same for all robots). The objective is to plan the movements of all robots, which result in the shortest worst-case time needed for all robots to evacuate.

1.1 Related work

Mobile agents are autonomous entities traveling within geometric or graph-modeled environments. Besides *mobility*, agents possess the ability to *perceive* the environment, *compute*, and *communicate* among themselves. They collaborate in order to perform tasks assigned to them. When agents operate in geometric environments (then they are usually called robots) their performance is measured by the geometric distance travelled, most often disregarding their computing, communicating and environment-perceiving activities.

When the geometric environment is not known in advance by the mobile robots, in many papers their task consisted in exploring the environment[1, 2, 13, 17]. The coordination of exploration between multiple robots has been mainly studied by the robotics community [10, 25, 26]. However even if the main objective assigned to the robots is different from exploration, often part of their activity is devoted to the recognition or mapping of the terrain and/or the position of the robots within it [20, 22, 24]. When the map of the environment is known to the robots, a lot of research was devoted to search games, when the searchers usually try to minimize the time to find an immobile or a moving hider [3, 4, 21]. The literature of the case of mobile fugitives, often known as cops and robbers or pursuit-evasion games is particularly rich [12, 15], with numerous variations related to the type of environment, speed of evasion and pursuit, robots visibility and many others [23]. The searching for a motionless point target in the simple environment presented in our paper has some similarities with the lost at sea problem, [16, 18], the cow-path problem [8, 9], and with the plane searching problem [5, 6].

The problem of evacuation has been studied for grid polygons from the perspective of constructing centralized evacuation plans, resulting in the fastest possible evacuation from the rectilinear environment [14]. Previously, [7] considered evacuation planning as earliest-arrival flows with multiple sources giving the first algorithm strongly polynomial in input/output size.

Evacuation in a distributed setting, when the mobile robots (know the simple environment but not the exit positions) has been recently asked in [11] for the case of a line. They proved that evacuation of multiple uniform agents is as hard as the cow-path problem. Evacuation of two robots without wireless communication was discussed with the research group of M. Yamashita during the visit of the second co-author at Kyushu University [19]. The discussion focused on laying the foundations for the lower bound presented in this paper and seeking ways to improve the respective upper bound. However, the main objective of our problem is to find a compromise between, on the one hand, spreading sufficiently the robots so that they can find the exit point fast in parallel, and, on the other hand, not to spread them too far so that, when one robot finds the exit, the escape route to it of the other robots is not too long.

1.2 Preliminaries

The environment is a disk of unit radius. The robots start their movement at the centre of the disk. We assume that the perception device of the robot permits to recognize a boundary point of the environment when the robot arrives there. Similarly, we assume that a robot recognizes the presence of other robots at the same position as well the fact that the robot is currently at the exit point. We also assume that the robots are labeled, i.e. they may execute different algorithms. Each such algorithm instructs the robot to make the moves with a speed not exceeding its maximal speed. In particular, the algorithm may ask the robot to move towards the centre of the disk or a chosen point on its boundary or to follow the boundary clockwise or counterclockwise. The movement may be changed when the perception mechanism allows the robot to acquire some knowledge about the environment (e.g. the exit point, boundary point, a meeting point with another robot). The robots are allowed to stay motionless at the same point. If A and B are points on the perimeter of the disk, by $\widehat{AB}$ we will denote arc from A to B in the clockwise direction and by AB we will denote the cord connecting A and B. The length of $\widehat{AB}$ will be denoted by $|\widehat{AB}|$ and the length of AB will be denoted by $|AB|$.

1.3 Outline and results of the paper

In Section 2 we consider the evacuation problem for two robots, while Section 3 analyzes the case of three robots. Section 4 proves tight asymptotic bounds for k robots. Each section is divided into two parts consisting of the analysis for the non-wireless and wireless models, respectively.

Complexity details corresponding to the three sections are provided in Table 1, for $k = 2, 3$,

Table 1. Results for $k = 2$ and $k = 3$.

Communication	*Upper bound*	*Lower bound*
Non-wireless $k = 2$:	~ 5.74 Theorem 1	~ 5.199 Theorem 2
Wireless $k = 2$:	$1 + \frac{2\pi}{3} + \sqrt{3} \sim 4.83$ Theorem 3	$1 + \frac{2\pi}{3} + \sqrt{3} \sim 4.83$ Theorem 4
Non-wireless $k = 3$:	$3 + \frac{2\pi}{3} \sim 5.09$ Theorem 8	~ 4.519 Theorem 5
Wireless $k = 3$:	~ 4.22 Theorem 6	~ 4.159 Theorem 7

and in Table 2, for large k.

This establishes a separation between the non-wireless and the wireless communication models. Details of all missing proofs can be found in the appendix.

Table 2. Results for large k.

Communication	*Upper bound*	*Lower bound*
Non-wireless:	$3 + \frac{2\pi}{k}$ Theorem 8	$3 + \frac{2\pi}{k} - O(k^{-2})$ Theorem 9
Wireless:	$3 + \frac{\pi}{k} + O(k^{-4/3})$ Theorem 10	$3 + \frac{\pi}{k}$ Theorem 11

2 Two Robots

Consider a disk centered at K. Two robots, say r_1, r_2, start at K moving with constant speed, say 1, searching for an exit located at an unknown point on the perimeter of the disk. In the sequel we prove upper and lower bounds for the two robot case in the non-wireless and wireless cases.

2.1 Non-wireless communication

Algorithm $\mathcal{A}_1$ indicates the robot trajectory for evacuation without wireless communication.

Algorithm $\mathcal{A}_1$ [for two robots without wireless communication]

1. Both robots move to an arbitrary point A on the perimeter.
2. At A the robots move along the perimeter of the disk in opposite directions; robot r_1 moves counter-clockwise and robot r_2 moves clockwise until one of the two robots, say r_1, finds the exit at B.
3. Now robot r_1 is at point B and r_2 is at point C (symmetric to B). Robot r_1 chooses a point D such that the length of the chord BD is equal to the length of the arc $\widehat{CD}$ and moves towards D.
4. Since the length of the chord BD is equal to the length of the arc $\widehat{CD}$, both robots arrive at D at the same time. Robot r_1 has knowledge about the location of the exit thus both robots can now follow the straight line DB and exit.

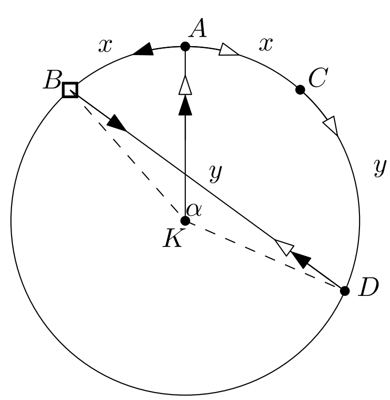

Fig. 1. Evacuation of two robots without wireless communication.

In the following theorem we give a bound on the worst-case evacuation time of algorithm $\mathcal{A}_1$.

Theorem 1. *There is an algorithm for evacuating the robots from an unknown exit located on the perimeter of the disk which takes time $1+\alpha/2+3\sin(\alpha/2)$, where the angle α satisfies the equation $\cos(\alpha/2) = -1/3$. It follows that the evacuation algorithm takes time ~ 5.74.*

Proof. (Theorem 1) We calculate the time required until both robots from algorithm $\mathcal{A}_1$ reach the exit. Denote $x = |\widehat{BA}| = |\widehat{AC}|$, $y = |BD| = |\widehat{CD}|$ and $\alpha = |\widehat{BD}|$. According to the definition of the above algorithm $\mathcal{A}_1$ the total time required is $f(\alpha) = 1 + x + 2y$. Observe that $\alpha = 2x + y$, and $y = 2\sin(\alpha/2)$, because y is a chord of the angle α. By substituting x and y in the definition of the function f we can express the evacuation time as a function of the angle α as follows. $f(\alpha) = 1 + \frac{\alpha-y}{2} + 2y = 1 + \frac{\alpha}{2} + \frac{3y}{2} = 1 + \frac{\alpha}{2} + 3\sin(\alpha/2)$. Now we differentiate with respect to α and we obtain: $\frac{df(\alpha)}{d\alpha} = \frac{1}{2} + \frac{3}{2}\cos(\alpha/2)$. It is easy to see that this derivative equals 0 for the maximum of function $f(\alpha)$, which yields as value for α the solution of $\cos(\alpha/2) = -1/3$. This proves Theorem 1.

We remark however that algorithm $\mathcal{A}_1$ is not optimal. We can introduce the following modification to the algorithm $\mathcal{A}_1$. Consider the trajectory of a robot until the robot neither had discovered the exit nor had been notified about the exit. In $\mathcal{A}_1$ the trajectory is radius KA and then starting from A, a semicircle (in some direction) of the perimeter. In the modified algorithm $\mathcal{A}_1'$ the trajectory is:

1. radius KA,
2. part of the semicircle of length z_1 to point E,
3. interval EF of length z_2 towards the center of the disk,
4. interval FE of length z_2 back to the perimeter,
5. remaining part of the semicircle.

When the robot is moving towards the center (item 3), the potential length y of the chord that needs to be traversed to get to the exit (if the exit is discovered by the other robot) is shorter than in the algorithm $\mathcal{A}_1$. We place the point E such that if the other robot discovered the exit in the worst case point then the robots will meet in the interior of the disk, not on the perimeter. Experiments showed that if $z_1 = 2.64$ and $z_2 = 0.5$ then the worst case evacuation time of the modified algorithm is 5.64. In the sequel we state and prove a lower bound.

Theorem 2. *It takes at least $3 + \frac{\pi}{4} + \sqrt{2}$ (~ 5.199) time units for two robots to evacuate from an unknown exit located in the perimeter of the disk.*

2.2 Wireless communication

Algorithm $\mathcal{A}_2$ indicates the robot trajectory for evacuation with wireless communication.

Algorithm $\mathcal{A}_2$ [for two robots with wireless communication]

1. Both robots move to an arbitrary point A on the perimeter.
2. At A the robots start moving along the perimeter of the disk in opposite directions: robot r_1 moves counterclockwise and robot r_2 moves clockwise until one of the robots, say r_1, finds the exit at B.
3. Robot r_1 notifies r_2 using wireless communication about the location of the exit and robot r_2 takes the shortest chord to B.

Theorem 3. *There is an algorithm for evacuating two robots from an unknown exit located on the perimeter of the disk which takes time at most* $1 + \frac{2\pi}{3} + \sqrt{3}$.

Proof. (Theorem 3) Consider the maximum evacuation time of algorithm $\mathcal{A}_2$. If the angular distance between A and B equals x, then the length of the chord taken by the robot r_2 equals to $c(x) = 2\sin(\pi - x)$ (see Figure 2). Thus the evacuation time T satisfies $T \leq \max_{0\leq x\leq\pi}\{1+x+2\sin(\pi-x)\} =$

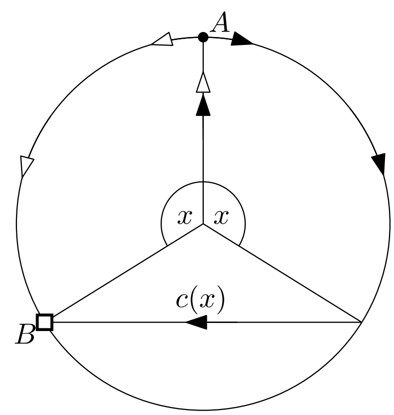

Fig. 2. Evacuation of two robots with wireless communication.

$\max_{0\leq x\leq\pi}\{1 + x + 2\sin x\}$. The function $f(x) = 1 + x + 2\sin x$ in the interval $[0, \pi]$ is maximized at the point $x^* = 2\pi/3$ and $f(x^*) = 1 + 2\pi/3 + \sqrt{3}$. This proves Theorem 3.

We now state the main lower bound.

Theorem 4. *For any algorithm it takes at least* $1 + \frac{2\pi}{3} + \sqrt{3}$ *time in the worst case for two robots to evacuate from an unknown exit located in the perimeter of the disk.*

3 Three Robots

In this section we analyze evacuation time for three robots in both non-wireless and wireless models.

3.1 Non-wireless communication

The first lemma provides a lower bound which is applicable for any k robots in the non-wireless model.

Lemma 1. *For any* $k \geq 3$ *and* $1 < \alpha < 2$*, it takes time at least* $\min\left\{3 + \frac{\alpha\pi}{k}, 3 + 2\sin\left(\pi - \frac{\alpha\pi}{2}\right)\right\}$ *in the worst case to evacuate from an unknown exit located on the perimeter of the disk in the model without wireless communication.*

Proof. (Lemma 1) Take any evacuation algorithm $\mathcal{A}$. Denote by $\mathcal{A}_r^p(t)$ the position of robot r in time t if the exit is located at point p. Since we are considering the worst case, we need to show that there exists a point p^* on the perimeter such that if the exit is located at p^* then the evacuation time of the algorithm $\mathcal{A}$ is at least $3 + \frac{2\pi}{k} - O(k^{-2})$. Consider the following three time intervals: $I_1 = [0,1), I_2 = \left[1, 1 + \frac{\alpha\pi}{k}\right), I_3 = \left[1 + \frac{\alpha\pi}{k}, 3\right)$. Since algorithm $\mathcal{A}$ is deterministic, the robots will follow a fixed trajectory, independent of the location of the exit until finding the exit or being notified about it by some other robot. Denote these trajectories by $p_1(t), p_2(t), \ldots p_k(t)$. Consider two cases:

Case 1. *There exists a robot r and time $t^* \in I_3$ such that point $p = p_r(t^*)$ of the trajectory of the robot r is on the perimeter of the disk.*

We will argue that the adversary can place the exit at point p^* being antipodal of p. We need to prove that if the exit is at point p^* then until time t^* robot r will be unaware of the location of the exit and will follow the trajectory $p_r(t)$. Consider the trajectory followed by robot r in algorithm $\mathcal{A}$ if the exit is at point p^*. Robot r is following the trajectory $p_r(t)$ until finding the exit or being notified about it. We want to show that robot r cannot be notified about the exit until time t^*. Assume on the contrary that $1 \leq t' < t^*$ is the first moment in time when r either discovered the exit or met a robot carrying information about the location of the exit. Thus we have that $\mathcal{A}_r^{p^*}(t) = p_r(t), \quad \text{for all } t \in [0, t']$. First note that since $p = p_r(t^*)$ we have that $dist(\mathcal{A}_r^{p^*}(t'), p^*) = dist(p_r(t'), p^*) > t' - 1$. The last inequality is true because if the distance between $p_r(t')$ and p^* would be at most $t' - 1$ then the distance to p would be at least $3 - t'$ (because p and p^* are antipodal) and robot r following trajectory $p_r(t)$ would not be able to reach p until time t^* (recall $t^* < 3$), which is a contradiction since $p_r(t^*) = p$. Now observe that in algorithm $\mathcal{A}$ if the exit is located at p^* then for any time moment $t' \leq 3$, any robot carrying information about the location of the exit is at distance at most $1 - t'$ from p^* (it is because robots can exchange informations only when they meet and the maximum speed of a robot is 1). Thus it is not possible that robot r in time t' obtain the information about the exit by meeting another robot. It is also not possible that $p_r(t') = p^*$, because robot r following trajectory $p_r(t)$ would not be able to reach p until time t^*. Thus such t' does not exist and we have: $\mathcal{A}_r^{p^*}(t) = p_r(t), \quad \text{for all } t \in [0, t^*]$. In time moment t^* robot r following algorithm $\mathcal{A}$ is at distance 2 from the exit located at p^*. Thus the total evacuation time is at least $t^* + 2 \geq 3 + \alpha\pi/k$, since $t^* \geq 1 + \alpha\pi/k$ (because $t^* \in I_3$).

Case 2. *None of the trajectories $p_1(t), p_2(t), \ldots p_k(t)$ in the interval I_3 is equal to a point on the perimeter.*

In this case we consider robots following the trajectories $p_1(t), p_2(t), \ldots, p_k(t)$ in the time interval $[0, 3)$. The set of points U on the perimeter of the disk that were not visited by any robot following such trajectories satisfies $|U| \geq 2\pi - \alpha\pi$ because in this case robots can explore the perimeter only in time interval I_2 of length $\alpha\pi/k$. Thus by Lemma 5 there exists a pair of unexplored points at distance at least $2\pi - \alpha\pi - \epsilon$ for any $\epsilon > 0$. The chord connecting these two points has length at least $2\sin(\pi - \alpha\pi/2 - \epsilon/2)$. Take this chord and denote its endpoints by u_1 and u_2. The adversary can run the algorithm $\mathcal{A}$ until moment t' when one of the points u_1, u_2 is visited and the adversary can place the exit in the other one. Note that until moment t' robots are following trajectories $p_r(t)$ because none of the robots has any information about the exit, thus $t' \geq 3$. Now the first robot that visited one of the points u_1, u_2 still needs to travel at least $2\sin(\pi - \alpha\pi/2 - \epsilon/2)$ because the exit is on the other end of the chord. Thus exploration time is in this case at least $3 + 2\sin(\pi - \alpha\pi/2 - \epsilon/2)$. We showed that the worst case time of evacuation T for any correct algorithm satisfies $T \geq \min\left\{3 + \frac{\alpha\pi}{k}, 3 + 2\sin\left(\pi - \frac{\alpha\pi}{2} - \frac{\epsilon}{2}\right)\right\}$, for any $\epsilon > 0$. The claim of the lemma follows by passing to the limit as $\epsilon \to 0$.

Theorem 5. *It takes at least* 4.519 *time in the worst case to evacuate three robots from an unknown exit located in the perimeter of the disk in the model without wireless communication.*

Proof. (Theorem 5) We have by Lemma 1 that the evacuation time T of any evacuation algorithm $\mathcal{A}$ satisfies $T \geq \min\{3 + \frac{\alpha\pi}{k}, 3 + 2\sin(\pi - \alpha\pi/2)\}$ for any $k \geq 3$. To prove the statement we numerically find such α that $\frac{\alpha\pi}{3} = 2\sin\left(\pi - \frac{\alpha\pi}{2}\right)$. If we set $\alpha = 1.408$, we obtain $T \geq \min\left\{3 + \frac{\alpha\pi}{3}, 3 + 2\sin\left(\pi - \frac{\alpha\pi}{2}\right)\right\} > 4.519$. This proves Theorem 5.

3.2 Wireless communication

We have three robots r_1, r_2, r_3 and consider the following algorithm.

Algorithm $\mathcal{A}_3$ [for three robots with wireless communication]

1. Robot r_1 moves to an arbitrary point A of the perimeter, robots r_2 and r_3 move together to the point B at angle $y = 4\pi/9 + 2\sqrt{3}/3 - 401/300$ in the clockwise direction to the radius taken by robot r_1.
2. Robot r_1 moves in the counter-clockwise direction. Robot r_2 moves in the clockwise direction. Robot r_3 moves in the counter-clockwise direction for time y. Then r_3 moves towards the center. Then r_3 moves towards the perimeter at angle $\pi - y/2$ in the clockwise direction to radius RB.
3. A robot that discovers the exit sends notification to other robots.
4. Upon receiving notification a robot walks to the exit using the shortest path.

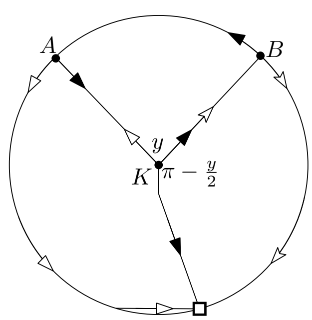

Fig. 3. Evacuation of three robots with wireless communication.

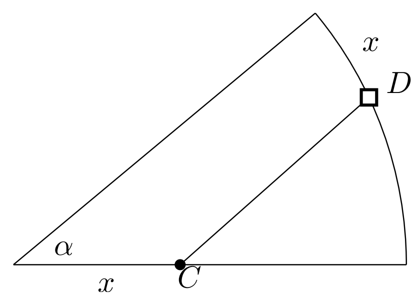

Fig. 4. $|CD| = \sqrt{1 - 2x\cos(\alpha - x) + x^2}$

The upper bound is proved in the following theorem.

Theorem 6. *It is possible to evacuate three robots from an unknown exit located on the perimeter of the disk in time at most* $\frac{4\pi}{9} + \frac{2\sqrt{3}+5}{3} + \frac{1}{600} < 4.22$ *in the model with wireless communication.*

The lower bound is proved in the following theorem.

Theorem 7. *Any algorithm takes at least* $1 + \frac{2}{3}\arccos\left(-\frac{1}{3}\right) + \frac{4\sqrt{2}}{3} \sim 4.159$ *time in the worst case for three robots to evacuate from an unknown exit located in the perimeter of the disk.*

4 *k* Robots

We prove asymptotically tight bounds for k robots in both the non-wireless and wireless models.

4.1 Non-wireless communication

The trajectory of the robots is given in algorithm $\mathcal{A}_4$.

Algorithm $\mathcal{A}_4$ [for k robots with wireless communication]

1. The k robots "spread" at equal angles $2\pi/k$ and they all reach the perimeter of the disk in time 1.
2. Upon reaching the perimeter, they all move clockwise along the perimeter for $2\pi/k$ time units.
3. In one time unit, all robots move to the center of the disk. Since at least one robot has found the exit it can inform the remaining robots.
4. In one additional time unit all robots move to the exit.

Theorem 8. *It is possible to evacuate k robots from an unknown exit located on the perimeter of the disk in time $3 + \frac{2\pi}{k}$ in the model with local communication.*

Proof. (Theorem 8) Clearly the algorithm $\mathcal{A}_4$ is correct and attains the desired upper bound.

The following technical lemma provides bounds on the sin and cos functions based on their corresponding Taylor series expansions.

Lemma 2. *For any $x \geq 0$ we have the following bound on values of $\sin x$ and $\cos x$:*

(1) $\sin x \geq x - x^3/3!$
(2) $\cos x \leq 1 - x^2/2! + x^4/4!$

Theorem 9. *It takes time at least $3 + \frac{2\pi}{k} + O(k^{-2})$ in the worst case to evacuate three robots from an unknown exit located on the perimeter of the disk in the model without wireless communication.*

For $k \geq 3$ robots we conjecture that the time T required to find a exit on the perimeter of a disk is exactly $3 + \frac{2\pi}{k}$.

4.2 Wireless communication

The trajectory of the robots is given in algorithm $\mathcal{A}_5$.

Algorithm $\mathcal{A}_5$ [for k robots with wireless communication]

1. Divide the team of robots into two groups: Group G_α of size $k_\alpha = \lceil k^{2/3} \rceil$, and Group G_β of size $k_\beta = k - k_\alpha$.
2. Assign a continuous arc $\widehat{AB}$ of length $\pi - 2\sqrt{\pi}k^{-1/3}$ to group G_α and remaining part of the perimeter denoted by $\widehat{BA}$ (of length $\pi + 2\sqrt{\pi}k^{-1/3}$) to group G_β.
3. Divide arcs $\widehat{AB}$ and $\widehat{BA}$ equally between members of groups. Each robot belonging to G_α is assigned an arc of length $a_\alpha = |\widehat{AB}|/k_\alpha$. Each robot from group G_β receives an arc of length $a_\beta = |\widehat{BA}|/k_\beta$.
4. Each robot goes from the center to the perimeter and explores an assigned arc. Extremal robots from group G_α when exploring the assigned arcs go towards each other (see Figure 5). All other robots explore assigned arcs is any direction. A robot that discovers the exit sends notification to all other robots using wireless communication.
5. Upon receiving a notification about the position of the discovered exit, a robot takes the shortest chord to the exit.
6. Robots from group G_β after finishing exploration of their arcs start moving towards the center.

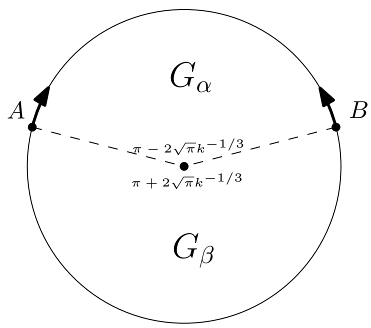

Fig. 5. Extremal (leftmost and rightmost) robots from group G_α are moving towards the interior of the arc $\widehat{AB}$.

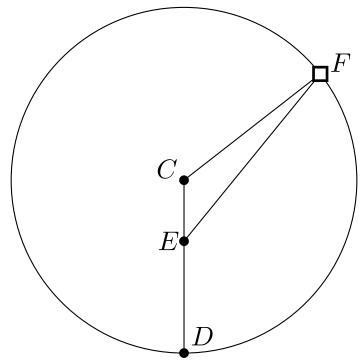

Fig. 6. $|DE| + |EF| < |DC| + |CF|$

Theorem 10. *If $k \geq 100$ then it is possible to evacuate k robots from an unknown exit located in the perimeter of the disk in time $3 + \frac{\pi}{k} + O(k^{-4/3})$, in the model with wireless communication.*

Proof. (Theorem 10) Consider the evacuation time of the algorithm $\mathcal{A}_5$. Note that since $k \geq 100$ then $k - \lceil k^{2/3} \rceil \geq \lceil k^{2/3} \rceil$ implying that $a_\alpha > a_\beta$. Thus robots from G_β finish exploration first and start going towards the center while robots from G_α are still exploring (point 6. in the pseudocode). We will show an upper bound on evacuation time T of the algorithm. Consider two cases:

Case 1. *The exit is located within the arc $\widehat{AB}$.*

Consider the evacuation time T_β of robots from group G_β. Observe that since $\epsilon > 1$, then $a_\alpha < 1$ thus the exit is discovered while robots from G_β are walking towards the center (before they reach the center). Robots from G_β start moving towards the center at time $1 + a_\beta$. At some time t' satisfying $2 + a_\beta > t' > 1 + a_\beta$ the exit is discovered by a robot from group G_α. Consider a trajectory taken by a robot r from group G_β starting from time $1 + a_\beta$. If r would simply walk to the center and then from the center to the exit (location of the exit would be known by the time when r reaches the center). The time would be $t' + 2$. By the triangle inequality the path taken by robot r acting according to the algorithm is shorter (see Figure 6). Thus the evacuation time T_β for robots belonging to team G_β is at most

$$\begin{aligned} T_\beta \leq t' + 2 &\leq 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k - k_\alpha} \\ &= 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k} + \frac{(\pi + 2\sqrt{\pi}k^{-1/3})\lceil k^{2/3} \rceil}{k(k - \lceil k^{2/3} \rceil)} = 3 + \frac{\pi}{k} + O(k^{-4/3}). \end{aligned}$$

Consider now the evacuation time of robots from group G_α. Assume that the exit is discovered at time $1 + x$ for some $0 \leq x \leq a_\alpha$. Since the extremal robots from group G_α are walking towards each other at the time moment $1 + x$ two arcs of length x has been explored starting from each endpoint of arc $\widehat{AB}$. Thus the distance on the perimeter between extremal unexplored points of arc $\widehat{AB}$ is $\pi - 2\sqrt{\pi}k^{-1/3} - 2x$. Hence the maximum length of a chord connecting two unexplored points of arc $\widehat{AB}$ in this moment is $2\sin((\pi - 2\sqrt{\pi}k^{-1/3} - 2x)/2)$. Therefore the time T_α until evacuation of all robots from group G_α is at most

$$T_\alpha \le \max_{0\le x\le a_\alpha} \left\{1 + x + 2\sin\left(\frac{\pi - 2\sqrt{\pi}k^{-1/3} - 2x}{2}\right)\right\}$$
$$= \max_{0\le x\le a_\alpha} \left\{1 + x + 2\cos\left(\sqrt{\pi}k^{-1/3} + x\right)\right\}.$$

The function $f(x) = 1 + x + 2\cos(\sqrt{\pi}k^{-1/3} + x)$ has derivative $f'(x) = 1 - 2\cos(\sqrt{\pi}k^{-1/3} + x)$. For $k \ge 100$ we have that $2\sqrt{\pi}k^{-1/3} + a_\alpha \le \pi/6$. Thus $\cos(\sqrt{\pi}k^{-1/3} + x) \le 1/2$ for all $x \in [0, a_\alpha]$, which implies that the function $f(x)$ is non-decreasing in the considered set. In order to find the maximum it is sufficient to consider its value at the extremal point a_α.

$$\begin{aligned}
T_\alpha &\le 1 + a_\alpha + 2\sin(\pi/2 - (\sqrt{\pi}k^{-1/3} + a_\alpha)) \\
&= 1 + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3}\rceil} + 2\cos\left(\sqrt{\pi}k^{-1/3} + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3}\rceil}\right) \\
&\le 1 + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3}\rceil} + 2 - \left(\sqrt{\pi}k^{-1/3} + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3}\rceil}\right)^2 + \left(\sqrt{\pi}k^{-1/3} + \frac{\pi - 2\sqrt{\pi}k^{-1/3}}{\lceil k^{2/3}\rceil}\right)^4/12 \\
&\le 3 + O(k^{-4/3})
\end{aligned}$$

Thus in this case the evacuation time $T \le \max\{T_\alpha, T_\beta\} \le 3 + \frac{\pi}{k} + O(k^{-4/3})$.

Case 2. *The exit is located within arc* $\widehat{BA}$.

Each robot from group G_β explores an arc of length $(\pi + 2\sqrt{\pi}k^{-1/3})/(k - k_\alpha)$. Thus time until the exit is discovered is at most $1 + (\pi + 2\sqrt{\pi}k^{-1/3})/(k - \lceil k^{2/3}\rceil)$. Since we are in the wireless communication model, each robot is notified immediately and needs additional time at most 2 to go to the exit. Thus the total evacuation time in this case is at most

$$\begin{aligned}
T &\le 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k - k^{2/3} - 1} \\
&= 3 + \frac{\pi + 2\sqrt{\pi}k^{-1/3}}{k} + \frac{(\pi + 2\sqrt{\pi}k^{-1/3})(k^{2/3} + 1)}{k(k - k^{2/3} - 1)} \\
&= 3 + \frac{\pi}{k} + O(k^{-4/3})
\end{aligned}$$

This completes the proof of Theorem 10.

Theorem 11. *It takes at least $3 + \frac{\pi}{k}$ time in the worst case to evacuate $k \ge 2$ robots from an unknown exit located on the perimeter of the disk in the model with wireless communication.*

5 Conclusion

We studied the evacuation problem for k robots in a disk of unit radius and provided several algorithms in both non-wireless and wireless communication models for $k = 2$ and $k = 3$ robots. For the case of k robots we were able to give asymptotically tight bounds thus indicating a clear separation between the non-wireless and the wireless communication models. There are many interesting open questions. An interesting challenge would be to tighten our bounds or even determine optimal algorithms for $k = 2, 3$ robots. Another interesting class of problems is concerned with evacuation from more than one exit, or with robots having distinct maximal speeds. Finally, the geometric domain being considered, the starting positions of the robots, as well as the communication model provide challenging variants of the questions considered in this paper.

References

1. S. Albers, M.R. Henzinger. Exploring Unknown Environments. SIAM J. Comput. 29(4): 1164-1188 (2000).
2. S. Albers and K. Kursawe and S. Schuierer, Exploring unknown environments with obstacles, Algorithmica 32, 123–143 (2002).
3. S. Alpern and S. Gal, The theory of search games and rendezvous. Int. Series in Operations research and Management Science, number 55, Kluwer Academic Publishers, 2002.
4. S. Alpern, R. Fokkink, L. Gasieniec, R. Lindelauf and V.S. Subrahmanian Eds, Search Theory, A Game Theoretic Approach, Springer, 2013.
5. R. A. Baeza-Yates, J.C. Culberson and G.J. E. Rawlins. Searching in the Plane. Inf. Comput. 106(2): 234-252 (1993).
6. R. A. Baeza-Yates and R. Schott. Parallel Searching in the Plane. Comput. Geom. 5: 143-154 (1995).
7. N. Baumann and M. Skutella, Earliest Arrival Flows with Multiple Sources. Math. Oper. Res. 34(2): 499-512 (2009).
8. A. Beck. On the linear search Problem, Naval Res. Logist. 2, 221-228, (1964).
9. R. Bellman. An optimal search problem, SIAM Rev. 5, 274, (1963).
10. W. Burgard, M. Moors, C. Stachniss and F.E. Schneider. Coordinated multi-robot exploration. IEEE Transactions on Robotics 21(3): 376-386 (2005).
11. M. Chrobak, L. Gasieniec, T. Gorry, and R. Martin, Evacuation problem on the line, in preparation.
12. T.H. Chung, G.A. Hollinger, V. Isler. Search and pursuit-evasion in mobile robotics. Autonomous Robots 31:4, 299-316 (2011).
13. X. Deng, T. Kameda and C. H. Papadimitriou, How to learn an unknown environment, Proc. 32nd Symp. on Foundations of Computer Science, 298–303, (1991).
14. S.P. Fekete, C. Gray, A. Kroller. Evacuation of Rectilinear Polygons. COCOA (1): 21-30, (2010).
15. F.V. Fomin, D.M. Thilikos. An annotated bibliography on guaranteed graph searching. Theoretical Computer Science 399:3, 236-245 (2008).
16. B. Gluss, An alternative solution to the "lost at sea?? problem, Naval Research Logistics Quarterly, 8(1), pp. 117-122, 1961.
17. F. Hoffmann. C. Icking, R. Klein and K. Kriegel, The polygon exploration problem, SIAM J. Comp. 31, 577–600 (2001).
18. J. Isbell, Pursuit Around a Hole, Naval Research Logistics Quarterly, 14(4), pp. 569-571, 1967.
19. S. Kijima, M. Yamashita, and Y. Yamauchi, private communication (2013).
20. J.M. Kleinberg, On-line Search in a Simple Polygon. SODA, 8-15, (1994).
21. T. Lidbetter, Hide-and-seek and Other Search Games, PhD Thesis, London School of Economics, 2013.
22. J.S.B. Mitchell, Geometric shortest paths and network optimization, Handbook of computational geometry, 633–702 (2000).
23. P. Nahin, Chases and Escapes: The Mathematics of Pursuit and Evasion, Princeton University Press, Princeton, NJ, 2007.
24. C.H. Papadimitriou, M. Yannakakis, Shortest paths without a map, Theor. Comput. Sci. 84, 127–150 (1991).
25. S. Thrun, A Probabilistic On-Line Mapping Algorithm for Teams of Mobile Robots. I. J. Robotic Res., 20(5):335-363 (2001).
26. B. Yamauchi, Frontier-Based Exploration Using Multiple Robots. Agents, 47-53 (1998).

A Lower bound in the non-wireless model for two robots

Proof. (Theorem 2) At the beginning, both robots are located at the center K of the disk. It takes at least 1 time unit for both of them to move to the perimeter of the disk.

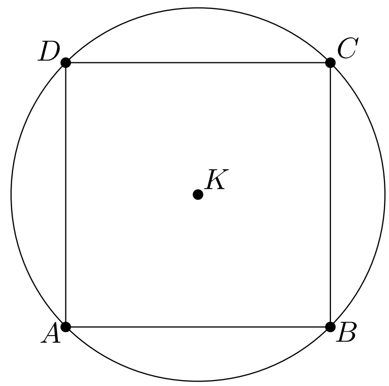

Fig. 7. Forming a square $ABCD$ of positions not yet explored by the robots.

In less than an additional $\pi/4$ time units the two robots cover at most a length of $\pi/2$ of the perimeter. The main idea is to observe, that until that time of the movement we can always construct a square $ABCD$ with sides equal to $\sqrt{2}$ whose all vertices are not yet visited by neither of the two robots. The vertices represent positions where an adversary can place an exit. Using an adversary argument it can be shown that an additional $2+\sqrt{2}$ time units are required for robot evacuation. We give details of this argument in the following two lemmas.

Lemma 3. *For any $\epsilon > 0$, at time $1+\frac{\pi}{4}-\epsilon$ there exists a square inscribed in the disk none of whose vertices has been explored by a robot.*

Proof. (Lemma 3) The proof is easily derived by rotating a square inscribed in the disk continuously for an angle of $\pi/2$. More precisely assume on the contrary that such an inscribed square does not exist. Consider a partition of perimeter of the disk into four arcs of length $\pi/2$, E_1, E_2, E_3, E_4. Any point $e_1 \in E_1$ uniquely defines an inscribed square with vertices $e_1 \in E_1, e_2 \in E_2, e_3 \in E_3, e_4 \in E_4$. Moreover for a different $e_1' \in E_1$, $e_1' \neq e_1$ vertices of the inscribed square $\{e_1', e_2', e_3', e_4'\}$ are different $e_i' \neq e_i$ for all $i \in 1,2,3,4$. By the assumption, for any $e_1 \in E_1$ at least one of the vertices $\{e_1, e_2, e_3, e_4\}$ of the inscribed square has to be explored (denote it by e^*). Thus for any e_1 we can identify an explored vertex $e^*(e_1)$. Since for different e_1, the inscribed square is different then the function $e^*(e_1)$ is an injection. Thus the image of the function $e^*(e_1)$ is a set of length $\pi/2$ of explored points. But such set does not exist because at time $1+\pi/4-\epsilon$ the total length of the set of explored points less than $\pi/2$. Therefore we obtain a contradiction at time $1+\frac{\pi}{4}-\epsilon$ that an inscribed square, none of whose vertices has been explored by a robot, does exist.

Lemma 4. *For any square inscribed in the disk none of whose vertices has been explored by a robot it takes more than $2+\sqrt{2}$ time to evacuate both robots from a vertex of the square.*

Proof. (Lemma 4) Take the square $ABCD$ with unexplored vertices. Consider any evacuation algorithm $\mathcal{A}$. We allow the algorithm to place the robots on arbitrary positions of the disk (possibly also on vertices of the square). The adversary can run the algorithm with undefined position of the exit and place the exit depending on the behaviour of the robots. The adversary will run the algorithm from perspective of a fixed robot r and will place the exit at a some point P. The placement of the exit at point P in time t is possible if robot r has no information whether the exit is located in P. Formally we say that a point P is *unknown* to robot r at time t if for any time moment $t' \in [0, t]$ robot r is at distance more than t' from P. This means that even if other robot started at P it could not meet r at any time in the interval $[0, t]$. Take a robot r and the first time moment t when the third vertex of the square is visited by a robot. Consider two cases

Case 1. $\sqrt{2} \leq t < 2$.

Denote the vertex visited by r in time t by A. The adversary places the exit in the antipodal point C. Observe that point C is unknown to r at time t. This is because if r was at distance at most t' from C at some time $t' \in [0, t]$ then it would be at distance $2 - t'$ from A and would reach A no sooner than at time 2, which is a contradiction as $t < 2$. Thus placement of the exit in C cannot affect movement of r until time t. Therefore, the adversary can place the exit in C and the evacuation time in this case will be at least $t + 2 \geq 2 + \sqrt{2}$.

Case 2. $2 \leq t$.

Time moment t is the first time when three vertices of the square are explored (it is possible that in t both robots explore a new vertex). Therefore, at time t, some robot r has knowledge about at most three vertices. The adversary simply places the exit in the vertex unknown to r and the evacuation time of r will be at least $t + \sqrt{2} \geq 2 + \sqrt{2}$.

Observe that t cannot be smaller than $\sqrt{2}$ because within time t at least one robot has to traverse at least one side of the square. This proves Lemma 4.

Clearly, the proof of Theorem 2 is an immediate consequence of Lemmas 3 and 4.

B Lower bound in the wireless model for two robots

In order to prove the lower bound we need to show the following lemma.

Lemma 5. *Consider a perimeter of a disk whose subset of total length $u + \epsilon > 0$ has not been explored for some $\epsilon > 0$ and $\pi \geq u > 0$. Then there exist two unexplored boundary points between which the distance along the perimeter is at least u.*

Proof. (Lemma 5) Denote by U set of all unexplored points. We have that $|U| = u + \epsilon$. First consider the case when $u < \pi$. Throughout the proof we will consider only points on the perimeter of the disk. Let $dist(x_1, x_2)$ be defined as the length of the shorter arc connecting x_1 and x_2.

Assumption 1*: Assume, on the contrary, that two unexplored boundary points between which the distance along the perimeter is at least u do not exist.*

Under such assumption we will construct subsets N, L, R of the set of all unexplored points ($N, L, R \subset U^{\mathsf{c}}$). Set N is defined as the set of all antipodal points of points in U, (if $x \in U$, then $y \in N$ if and only if $dist(x, y) = \pi$). For any $x \in U$, by $x + \pi$ we denote the point antipodal to x. To construct L and R take any $x \in U$. Let x' and x'' be the unexplored point closest to $x + \pi$ in the clockwise and counter-clockwise direction respectively. We construct arc L as the set of points on the perimeter at distance at most $\pi - u$ from $x' + \pi$ (antipodal to x') in the counter-clockwise direction. Similarly, R is the set of points at distance at most $\pi - u$ from $x'' + \pi$ in the clockwise direction (see Figure 8).

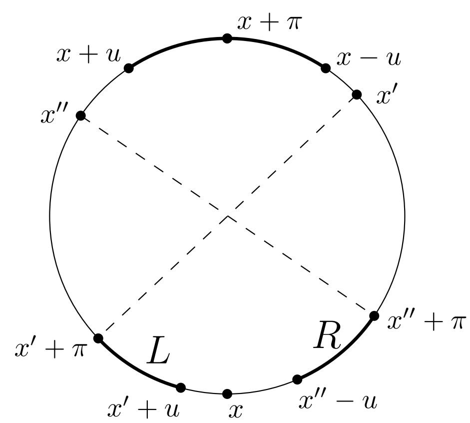

Fig. 8. Construction of sets L and R.

Observe that all points belonging to sets N, L, R are explored. Every point $y \in N$ is antipodal to some unexplored point $y + \pi \in U$, thus if y is unexplored then we have a pair of unexplored points y, $y + \pi$ at distance π. If a point y' in L is unexplored then we have a pair of unexplored points x', y' at distance at least u. Finally if a point y'' in R is unexplored then we have a pair of unexplored points x'', y'' at distance at least u. All these cases lead do contradiction with Assumption 1.

We want to show that $|L \cup R| = 0$. First note that

$$dist(x + \pi, x') > \pi - u, \tag{1}$$

because if $dist(x+\pi, x') \leq \pi - u$, then $dist(x, x') \geq u$ which is impossible due to Assumption 1 since both x and x' are unexplored. Similarly we observe that $dist(x + \pi, x'') > \pi - u$. By equation (1) we have that $dist(x' + \pi, x) = dist(x', x + \pi) > \pi - u$ thus set L is a subset of the semicircle from x to $x + \pi$ in the clockwise direction. Similarly we show that R is a subset of the semicircle from x to $x + \pi$ in the counter-clockwise direction. Thus $L \cup R$ contains at most one point (in the case when $x = x' = x''$). Thus $|L \cup R| = 0$.

Observe also that $|L \cup N| = 0$, because all points in the arc from $x + \pi$ to x' in the clockwise direction are explored (x' is the closest unexplored). Similarly $|R \cup N| = 0$.

Thus $|N \cup L \cup R| = |N| + |L| + |R| = u + \pi - u + \pi - u = 2\pi - u$. Since all points in N, L and R are explored we have $|U| = 2\pi - |U^{\mathsf{c}}| \leq 2\pi - |N \cup L \cup R| = u$ which is a contradiction because $|U| > u$. If $u = \pi$ it is sufficient to consider set N. Observe that all elements from set N are explored and $|N| = \pi$. We obtain contradiction because $|U| > \pi$.

Lemma 6. *For any $k \geq 2$ and x satisfying $\pi/k \leq x < 2\pi/k$ and any evacuation algorithm it takes time at least $1 + x + 2\sin(xk/2)$ to evacuate from an unknown exit located in the perimeter of the disk.*

Proof. (Lemma 6) Consider an algorithm $\mathcal{A}$ whose evacuation time equals to T. In any evacuation algorithm using k robots, at time moment $1 + x$, the total length of explored arcs of the perimeter equals at most xk (because robots need time 1 to go from the center to the perimeter). Thus the total length of the unexplored part of the perimeter is at least $2\pi - xk$ and $\pi \geq 2\pi - xk > 0$. Thus using Lemma 5 at time moment $1 + x$ there exists a pair of unexplored points whose distance on the perimeter is at least $2\pi - xk - \epsilon$ for any $\epsilon > 0$. Take this pair of points and consider a chord connecting them. Such chord has length at least $2\sin(\pi - xk/2 - \epsilon/2)$ and has both endpoints unexplored. Thus the adversary can place the exit in any of two endpoints. Consider the moment when some robot visits the first endpoint of the chord. The adversary places the exit in the other endpoint and such robot will have to walk at least the length of the chord. Thus the total evacuation time is at least $1 + x + 2\sin(\pi - xk/2 - \epsilon/2)$. This holds for any $\epsilon > 0$, thus by taking the limit $\epsilon \to 0$ we obtain $T \geq 1 + x + 2\sin(xk/2)$.

Proof. (Theorem 4) The theorem is a direct consequence of Lemma 6 by taking $k = 2$ and $x = 2\pi/3$.

C Bounds in the wireless model for three robots

Proof. (Theorem 6) Consider the evacuation time of the algorithm $\mathcal{A}_3$. If the exit is discovered within time $1+y$, then since we are working in the wireless communication model, time for evacuation is at most 2 after the discovery. Thus if the discovery is within time $1 + y$, the evacuation is in time at most $3 + y$. If the exit is discovered after time $1 + y$ then it is discovered either by r_1 or r_2 (robot r_3 explores part of the perimeter of length y thus he finishes exploration in time $1 + y$).

Consider the evacuation time of r_1 if the exit is discovered by r_2. Robot r_1 explores an assigned arc until being notified and upon notification he takes the chord to the exit. If the exit is discovered at time $1+y'$ then the evacuation time of r_1 is $T_{r_1} = 1+y'+2\sin(\pi-y/2-y')$, and $y' \in [0, \pi-y/2]$. In this interval the function $f(y') = 1+y'+2\sin(\pi-y/2-y')$ is maximized when $y' = 2\pi/3-y/2$ and the maximum value is $1+2\pi/3-y/2+\sqrt{3}$. Thus we have $T_{r_1} \leq 1+\frac{2\pi}{3}-\frac{y}{2}+\sqrt{3} = \frac{4\pi}{9}+\frac{2\sqrt{3}+5}{3}+\frac{1}{600}$. The evacuation time of r_2 can be bounded similarly.

Consider the evacuation time of r_3. Take the case when the exit is discovered by r_1 or r_2 in at some point of time in the interval $[1 + y, 2 + y]$. In this interval, robot r_3 is moving towards the center. A path from the point A to the center and from the center to the exit has length 2 (twice the radius). A path taken by the robot is shorter by the triangle inequality, because the robot after the discovery of the exit is not continuing to the center but it goes to the exit using the shortest path. Thus if the exit is discovered within interval $[1 + y, 2 + y]$ then the evacuation time of r_3 is at most $T_{r_3} \leq 3 + y = \frac{4\pi}{9} + \frac{2\sqrt{3}}{3} + \frac{998}{600} < \frac{4\pi}{9} + \frac{2\sqrt{3}+5}{3} + \frac{1}{600}$. Finally consider the evacuation time of robot r_3 in the case when the exit is discovered after time $2 + y$. In this case the exit is discovered while robot r_3 is walking from the center towards the perimeter. If the exit is discovered at time $2 + y + x$ then robot r_3 has walked distance x from the center. The length of the segment he takes to the exit equals $\sqrt{1 - 2x\cos(\alpha - x) + x^2}$ (see Figure 2), where α is length of the arc traversed by r_2 (or equivalently by r_1) after time $2 + y$. At time $2 + y$ the total length of the explored perimeter equals $3y + 2$. Thus $\alpha = \pi - 3y/2 - 1$. Thus the evacuation time of r_3 is in this case at most $T_{r_3} \leq 2 + y + \max_{x\in[0,\alpha]}\{x + \sqrt{1 - 2x\cos(\alpha - x) + x^2}\}$ We have that $\alpha = \pi - 3y/2 - 1 = \pi/2 - \sqrt{3} + 201/200 < 1/3$. In the interval $[0, 1/3]$ the cos function is decreasing thus $-2x\cos(1/3 - x) \geq -2x\cos(\alpha - x)$ for any $x \in [0, 1/3]$, thus we have

$$\begin{aligned}
\max_{x\in[0,\alpha]} \left\{x + \sqrt{1 - 2x\cos(\alpha - x) + x^2}\right\} &\leq \max_{x\in[0,1/3]} \left\{x + \sqrt{1 - 2x\cos(\alpha - x) + x^2}\right\} \\
&\leq \max_{x\in[0,1/3]} \left\{x + \sqrt{1 - 2x\cos(1/3 - x) + x^2}\right\}
\end{aligned}$$

To complete the proof we show the following

Claim: $x + \sqrt{1 - 2x\cos(1/3 - x) + x^2} \leq 1.005$ *for every* $x \in [0, 1/3]$

First we set the variable $z = 1/3 - x$. We have

$$\max_{x\in[0,1/3]} \left\{ x + \sqrt{1 - 2x\cos\left(\frac{1}{3} - x\right) + x^2} \right\} = \max_{z\in[0,1/3]} \left\{ \frac{1}{3} - z + \sqrt{1 - 2\left(\frac{1}{3} - z\right)\cos z + \left(\frac{1}{3} - z\right)^2} \right\}$$

Now using lemma 2 we have

$$\begin{aligned} \frac{1}{3} - z + \sqrt{1 - 2\left(\frac{1}{3} - z\right)\cos z + \left(\frac{1}{3} - z\right)^2} &\leq \frac{1}{3} - z + \sqrt{1 - 2\left(\frac{1}{3} - z\right)\left(1 - \frac{z^2}{2}\right) + \left(\frac{1}{3} - z\right)^2} \\ &= \frac{1}{3} - z + \frac{2}{3}\sqrt{1 + 3z + 3z^2 - \frac{9}{4}z^3} \end{aligned}$$

In order to prove that $\frac{1}{3} - z + \frac{2}{3}\sqrt{1 + 3z + 3z^2 - \frac{9}{4}z^3} \leq 1.005$ it is equivalent to show that $1 + 3z + 3z^2 - \frac{9}{4}z^3 \leq \frac{162409}{160000} + \frac{1209}{400}z + \frac{9}{4}z^2$, because for $z \in [0, 1/3]$, $1 + 3z + 3z^2 - \frac{9}{4}z^3 > 0$. Thus we need to show that $0 \leq z^3 - \frac{z^2}{3} + \frac{z}{100} + \frac{1}{150} + \frac{1}{40000}$. The polynomial $z^3 - \frac{z^2}{3} + \frac{z}{100} + \frac{1}{150} + \frac{1}{40000}$ in the interval $[0, 1/3]$ has the minimal value for $z = 1/9 + \sqrt{73}/90$, and the minimal value is larger than 0. This finishes the proof of the claim.

Using the claim we have that in the case when exit is discovered after time $2+y$, the evacuation time T_{r_3} of robot r_3 satisfies $T_{r_3} \leq 2 + y + 1.005 = \frac{4\pi}{9} + \frac{2\sqrt{3}+5}{3} + \frac{1}{600}$. We bounded the evacuation time of each robot in every possible position of the exit thus the evacuation time T of the algorithm satisfies $T \leq \frac{4\pi}{9} + \frac{2\sqrt{3}+5}{3} + \frac{1}{600}$.

Proof. (Theorem 7) The theorem is a direct consequence of Lemma 6 by taking $k = 3$ and $x = 2/3\arccos(-1/3)$.

Proof. (Lemma 2) In the proof we assume that $x \geq 0$. We have: $f(x) = \sin x - x + \frac{x^3}{3!}, f'(x) = \cos x - 1 + \frac{x^2}{2}, f''(x) = -\sin x + x, f'''(x) = -\cos x + 1$ We have that $f(0) = f'(0) = f''(0) = f'''(0) = 0$. Since $\cos x \geq 1$, we have $f'''(x) \geq 0$. Thus $f''(x)$ is non-decreasing and since $f''(0) \geq 0$ then $f''(x) \geq 0$. Similarly we show that $f'(x) \geq 0$ and finally that $f(x) \geq 0$. Thus $\sin x \geq x - x^3/3!$ which proves (1). To prove (2) observe that: $g(x) = \cos x - 1 + \frac{x^2}{2!} - x^4/4!, g'(x) = -\sin x + x - \frac{x^3}{3!} = -f(x)$ Thus $g'(x) \leq 0$ and $g(x)$ is non-increasing. Now since $g(0) = 0$ we have $g(x) \leq 0$.

D Bounds in the wireless model for k robots

Proof. (Theorem 9) We have by Lemma 1 that the evacuation time T of any evacuation algorithm $\mathcal{A}$ satisfies $T \geq \min\{3 + \frac{\alpha\pi}{k}, 3 + 2\sin(\pi - \alpha\pi/2)\}$. If we set $\alpha = 2k/(k+1)$ then taking into account Lemma 2 we obtain:

$$\begin{aligned} T \geq \min\left\{3 + \frac{\pi}{k+1}, 3 + 2\sin\left(\frac{\pi}{k+1}\right)\right\} &\geq 3 + \frac{\pi}{k+1} - \frac{\pi^3}{3!(k+1)^3} \\ = 3 + \frac{\pi}{k} - \frac{\pi}{k(k+1)} - \frac{\pi^3}{3!(k+1)^3} &= 3 + \frac{\pi}{k} - O(k^{-2}), \end{aligned}$$

This proves the theorem.

Proof. (Theorem 11) The theorem is a direct consequence of Lemma 6 by taking $x = \pi/k$.