Lower Bounds for Compact Routing
(Extended abstract)

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Abstract

In this paper we present lower bounds for compact routing schemes. We give (1) networks on n vertices which for any interval routing scheme, $\Omega(n)$ routers of the network require $\Omega(n)$ intervals on some out-going link and (2) for each $d \geq 3$, networks of maximal degree d which for any interval routing scheme, $\Omega(n)$ routers each require $\Omega(n/\log n)$ intervals on some out-going link. Our results give the best known worst-case lower bounds for interval routing. For the case of universal routing schemes we give (3) networks on n vertices which for any near optimal routing scheme with stretch factor < 2 a total of $\Omega(n^2)$ memory bits are required, and (4) for each $d \geq 3$, networks of maximal degree d for which any optimal (resp., near optimal) routing scheme (resp., with stretch factor < 2) requires a total of $\Omega(n^2/\log n)$ (resp. $\Omega(n^2/\log^2 n)$) memory bits.

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1 Introduction

One of the most important measures of complexity of a routing scheme is the size of the routing information that must be stored locally and globally

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in the network. In general, in a universal routing scheme we are intersested in the number of bits stored, while in an interval routing scheme we are interested in the total number of intervals stored by the routers.

1.1 Interval routing model

In an interval labeling scheme of an n vertex network, node labels belong to the set $\{1, 2, \ldots, n\}$ and link labels are pairs of node labels representing disjoint intervals (possibly) with wrap-around. Such a scheme represents only one shortest path from the router to any node in the interval. Interval routing has been very successful in reducing the space requirements for routing in many standard networks, like rings, tori, hypercubes, etc. [14, 15, 11], as well as outerplanar [6], etc. In general, it is easy to see that an interval routing scheme can be implemented on any n vertex network with at most $O(n)$ intervals per router, hence a total of $O(n^2)$ intervals.

There have been several papers studying interval routing, e.g., [14, 15, 11, 6, 9, 10]. Kranakis, Krizanc and Ravi [11] considered lower bounds on the number of intervals needed per router and gave networks that have a router requiring $\Omega(n^{1/3})$ intervals at some link. Gavoille and Guévremont [10] proved an $\Omega(n)$ lower bound for a single router on a degree $\Theta(n)$ graph. Flammini, van Leeuwen and Marchetti-Spaccamela [4] proved that for an appropriately chosen class of (unbounded degree) graphs, with probability exceeding $1 - o(1)$, a randomly chosen graph contains a router requiring $\Omega(n^{1-1/\log^{1/2} n})$ intervals. For bounded degree graphs, Gavoille and Guévremont [9] show the existence of a degree 3 graph requiring $\Omega(n/log^2 n)$ interval at some router and and Braune [1] presents a degree 3 graph for which a single router requires $\Omega(n/\log n)$ intervals.

In this paper we prove optimal worst-case lower bounds for interval routing. We prove the existence of graphs satisfying the following conditions:

# Intervals	# Routers	# edges	Degree
$\Omega(n)$	$\Omega(n)$	$\Theta(n^2)$	$\Theta(n)$
$\Omega\left(\frac{n}{\log n}\right)$	$\Omega(n)$	$\Theta(n)$	$d \geq 3$

The first column represents a lower bound on the number of intervals required on a link of a router of the network and the second column is the number of routers guaranteed to have this property. The fourth column is the maximal degree of the network.

1.2 Universal routing model

The routing model we use for our lower bounds is the model of Peleg and Upfal [13]. A routing function $\mathcal{RF}$ is a triple (I, H, P) consisting of an *initialization, header*, and *port* functions, respectively. For any two distinct vertices u, v, $\mathcal{RF}$ produces a walk $u_0 := u, u_1, \ldots, u_k := v$ of vertices and a sequence $h_0, h_1, \ldots, h_k := v$ of headers such that $h_0 = I(u, v), P(u_k, h_k) = \emptyset$, and for all $i < k$, $H(u_i, h_i) = h_{i+1}, P(u_i, h_i) = (u_i, u_{i+1})$. A routing scheme is a distributed algorithm that computes a routing function for a given network; it is called near optimal with stretch factor s if the length of each of the walks it generates is at most s times the optimal length path. As in [13] the total memory requirement of a routing scheme is the number of memory bits required to store the functions I, H, and P, as well as the names given to the nodes of the network.

The hierarchical routing schemes of Peleg and Upfal [13] require storing a total of $O(n^{1+\frac{1}{s}})$ bits of routing information, for near-optimal routing schemes with strech factor $O(s)$, where $s \geq 1$ is fixed. In the same paper they also give an $\Omega(n^{1+1/(2s+4)})$ lower bound for such near optimal routing schemes. For planar networks, Frederickson and Janardan [7, 8] give near optimal routing schemes with stretch factor < 7 such that for all $\epsilon < 1/3$ the total memory requirement of the network is $O(n^{1+\epsilon} \log n/\epsilon)$ bits and the processor names are $O(\log n/\epsilon)$ bits long. For Kolmogorov random graphs, Buhrman, Hoepman and Vitányi [2] show how to do near optimal routing with stretch factor 2 with a total of $O(n \log n)$ memory bits for storing processor names plus $O(n \log\log n)$ for storing routing information.

Fraigniaud and Gavoille in a recent paper [5] proved an $\Omega(n^2)$ lower bound for near optimal universal routing schemes with stretch factor < 2 for a $\Theta(n)$ degree graph. As a result of our method we give a new simpler proof of this result using Kolmogorov complexity. We also extend this to near optimal universal routing schemes with stretch factor < 2, for bounded degree graphs.

For near optimal routing, we prove the existence of graphs satisfying the following conditions:

# Bits	# Edges	Stretch	Degree
$\Omega(n^2)$	$\Theta(n^2)$	$s < 2$	$\Theta(n)$
$\Omega\left(\frac{n^2}{\log n}\right)$	$\Theta(n)$	$s < 2$	$\Theta\left(\frac{n}{\log n}\right)$
$\Omega\left(\frac{n^2}{\log n}\right)$	$\Theta(n)$	$s = 1$	$d \geq 3$
$\Omega\left(\frac{n^2}{\log^2 n}\right)$	$\Theta(n)$	$s < 2$	$d \geq 3$

The first column represents a lower bound on the total memory requirements of the network, represented by the number of bits and the fourth column is the maximal degree of the network.

1.3 Outline of the paper

In section 2 we give the lemmas from Kolmogorov complexity that will be necessary for the remaining sections and also remind the reader of the technique of matrix of constraints (introduced by [4, 5]) which is crucial for the lower bounds we obtain. In section 3 we define the network and give lower bounds for near optimal and interval routing schemes. In section 4 we construct networks with bounded degree.

2 Preliminaries

2.1 Kolmogorov Complexity

In this section we apply the technique of Kolmogorov complexity to prove two lemmas that will be used in the sequel for the study of interval routing.

Consider a Kolmogorov random string M consisting of n^2 bits, i.e.

$$K(M) = n^2 + O(\log n). \tag{1}$$

In the sequel it will be convenient to think of M as an $n \times n$ matrix $M = (m_{i,j})$; this will enable us to refer to rows and columns of M in the obvious way.

LEMMA **1** *For any permutation of the rows (respectively, columns) of the matrix M the resulting matrix has $\Omega(n)$ columns (respectively, rows) each of which has Kolmogorov complexity $\Omega(n)$.*

PROOF By assumption $K(M) = n^2 + O(\log n)$. Let $M = [m_1, \ldots, m_n]$ denote the successive rows of the matrix listed from top to bottom. For a permutation π let $M^\pi = [m_{\pi(1)}, \ldots, m_{\pi(n)}]$. Since we can encode M from the permutation π and the permuted matrix M^π it follows that $K(M^\pi) \geq n^2 - n\log n + \Theta(\log n)$. Let K_j denote the Kolmogorov complexity of the jth column of the matrix M^π. Since the matrix M^π can be reconstructed from a description of its columns, it follows that

$$\sum_{j=1}^{n} K_j \geq K(M^\pi) - n\log n \in \Omega(n^2).$$

Let c be a constant such that $\sum_{j=1}^{n} K_j \geq cn^2$ and let J be the set of indices j such that $K_j \geq cn/2$. We have that

$$\sum_{j\in J}^{n} K_j + cn^2/2 \geq \sum_{j\in J}^{n} K_j + |J^c|cn/2 \geq$$

$$\sum_{j\in J}^{n} K_j + \sum_{j\notin J}^{n} K_j \geq cn^2.$$

It follows that

$$|J| \geq \sum_{j\in J}^{n} K_j \geq cn^2/2.$$

This last inequality implies that $K_j = cn/2$, for at least $cn/2$ columns j. This proves the lemma. ■

LEMMA **2** *For any permutation of the rows (respectively, columns) of the matrix M the resulting matrix has $\Omega(n)$ columns (respectively, rows) each of which has $\Omega(n)$ occurrences of the block* 01.

PROOF By Lemma 1 there exist $\Omega(n)$ columns (respectively, rows) each of which has Kolmogorov complexity $\Omega(n)$. By [12][Theorem 2.15, page 131] the number of occurrences of the block 01 in these columns is at least $\Omega(n) - \Theta(\sqrt{n}) \in \Omega(n)$. This completes the proof of the lemma. ■

The results extend to arbitrary $p\times q$ matrices such that $p, q \in \Omega(\log(pq))$. Let $P = (p_{i,j})$ be a Kolmogorov random $p \times q$ matrix satisfying $K(P) = pq + O(\log(pq))$.

LEMMA **3** *For any permutation of the rows (respectively, columns) of the matrix P the resulting matrix has $\Omega(q)$ columns (respectively, rows) each of which has Kolmogorov complexity $\Omega(p)$, and hence also $\Omega(p)$ occurrences of the block* 01. ■

2.2 Matrices of Constraints

Our lower bound technique is based on the matrix of s-constraints of a graph G developed by [4] and extended by [5]. A matrix of s-constraints for G is a $p \times q$ boolean matrix $P = (p_{i,j})$ whose rows are labeled with a subset $v_1, v_2, \ldots, v_p$ of the set of vertices and its columns by a subset $e_1, e_2, \ldots, e_q$ of the set of edges such that

- $p_{i,j} = 1 \Leftrightarrow$ "every near optimal shortest path with stretch factor s from the tail of e_j to v_i uses arc e_j", and
- $p_{i,j} = 0 \Leftrightarrow$ "no near optimal shortest path with stretch factor s from the tail of e_j to v_i uses arc e_j".

A matrix of constraints is the special case of a matrix of s-constraints, when $s = 1$. Let $R(P, e_j)$ be the number of occurrences of blocks of 01 in the column e_j of P. Then the following result is proved in [4].

LEMMA **4** *If P is a matrix of constraints of the graph G then for any interval routing scheme the number of intervals required by the router on the tail of edge e_j is at least $\min_\pi R(P^\pi, e_j)$, the minimum taken over all row permutations of P.* ∎

We can use the matrix of s-constraints to obtain lower bounds for near optimal routing schemes with stretch factor s. Let $P = (p_{i,j})$ be a Kolmogorov random $p \times q$ matrix satisfying $K(P) = pq + O(\log(pq))$, where $p, q \in \Omega(\log(pq))$. We can prove the following result.

LEMMA **5** *If P is a Kolmogorov random $p \times q$ matrix of s-constraints of the graph G then the total number of memory bits required by all the routers is bounded below by $\Omega(pq)$.*

PROOF (OUTLINE) Recall the definition of a routing schemes memory requirements includes the processor names in the the memory. Hence, we can assume without loss of generality that the sum of the number of bits used by all the processor names of the nodes $\{v_1, v_2, \ldots, v_p\}$ and the tails of the links $\{e_1, e_2, \ldots, e_q\}$ is $O(pq)$. (If that were not true the result would be obvious.) Assume on the contrary that we have a near optimal routing function with stretch factor s, say $\mathcal{RF}$, whose total memory bits required is $O(pq)$. Then it is easy to see that we can use the routing information supplied by the routing function $\mathcal{RF}$ on the tails of the vertices $e_1, e_2, \ldots, e_q$ with destination the

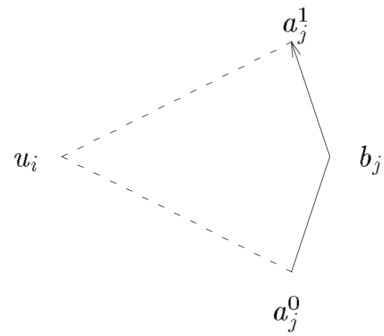

Figure 1: The (i,j)th component of the network G_M.

vertices $v_1, v_2, \ldots, v_q$, in order to reconstruct the matrix P of s-constraints. The extra overhead used is bounded from above by

$$\sum_{i=1}^{p} |name(v_i)| + \sum_{j=1}^{q} |name(tail(e_j))|.$$

This contradicts the fact that the matrix P is incompressible. We note that the constants involved in the previous arguments are easily seen to depend only on the definition of incompressibility [12]. ∎

3 Unbounded Degree Networks

In this section we give the lower bound proofs for near optimal and interval routing. We construct the network and subsequently prove the lower bounds for interval routing and near optimal routing.

Let $M = (m_{i,j})$ be a Kolmogorov random $n \times n$ matrix satisfying (1). The network we construct, denoted by G_M, is based on the construction of Fraigniaud and Gavoille [5] (depicted in Figure 1) and consists of the following:

1. A collection of $4n$ vertices consisting of
   - an independent set $u_1, u_2, \ldots, u_n$,
   - a set $a_1^b, a_2^b, \ldots, a_n^b$, where $b = 0, 1$, and

- a set $b_1, b_2, \ldots, b_n$.

A collection of n copies of a three vertex chain L_3; the jth copy consists of the vertices a_j^0, b_j, a_j^1 and the edges $\{a_j^0, b_j\}, e_j := (b_j, a_j^1)$, $j = 1, 2, \ldots, n$ (note we assume that the edge $e_j = (b_j, a_j^1)$ is directed).

2. For each $i, j = 1, 2, \ldots, n$ and $b = 0, 1$,

$$\{u_i, a_j^b\} \text{ is an edge } \Leftrightarrow m_{i,j} = b.$$

Clearly, the network G_M has $4n$ vertices and $\Theta(n^2)$ edges.

THEOREM **6** *G_M is a connected network of diameter* 4.

PROOF (OUTLINE) Our proof uses the randomness of the matrix M in an essential manner. It is easy to see that the result follows from the following claim.

Claim: For any a, b there exists a c such that $m_{a,c} = m_{b,c}$ and $m_{c,a} = m_{c,b}$.
PROOF of Claim. We prove the first part of the claim. The second part is similar. Let a, b be given indices. Assume on the contrary that for all c, $m_{a,c} \neq m_{b,c}$. This means we can reconstruct the ath row of the matrix from its bth row. Hence, we can encode the matrix M as follows:

- the $2 \log n$ bits representing the indices a, b, and
- the matrix M minus its ath row.

The number of bits in the above description is at most $n^2 - n + O(\log n)$, contradicting the incompressibility of the matrix M. ■

3.1 Interval routing

In the sequel we prove a lower bound on interval routing schemes.*

THEOREM **7** *In the n vertex network G_M, for every interval routing scheme there exist $\Omega(n)$ routers each requiring at least $\Omega(n)$ intervals in at least one of its links.*

*A simpler graph would be sufficient for the case of interval routing. For example, we could replace the length 3 chain in Figure 1 with the directed edge (a_j^0, a_j^1). The lower bound proof is the same. We chose the same graph as for the case of universal near optimal routing in order to shorten the overall proof.

PROOF We use the idea of matrix of constraints from subsection 2.2. The main theorem follows from Lemmas 2 and 4 and the Lemma below.

LEMMA **8** *M is a matrix of constraints for the graph G_M.*

PROOF Lemma 10 implies that M is a matrix of s-constraints for the graph G_M, and hence it is also a matrix of constraints for the same graph. This completes the proof of the Lemma and hence also of the theorem. ■

3.2 Near optimal routing

The same network can also be used for near optimal routing. We have a simpler proof of the following theorem, first proved in [5].

THEOREM **9** *In the n vertex network G_M, every near optimal routing scheme with stretch factor < 2 requires a total of $\Omega(n^2)$ memory bits.*

PROOF The main theorem follows easily from Lemma 5 and the Lemma below which was first proved by Fraigniaud and Gavoille [5].

LEMMA **10** *M is a matrix of s-constraints for the graph G_M, for any $s < 2$.*

PROOF For the given network G_M we consider the matrix whose rows are indexed by the vertices $u_1, u_2, \ldots, u_n$ and columns are indexed by the directed edges $e_j = (b_j, a_j^1)$, $j = 1, 2, \ldots, n$. The lemma follows easily since if a routing function uses link (b_j, a_j^1) either when it should not or it does not when it should then the resulting walk must have length ≥ 4. Since the optimal path has length 2 the stretch factor of the resulting path will be ≥ 2. It follows that the universal routing scheme encodes the above matrix M of constraints. ■

4 Bounded Degree Networks

In this subsection we construct bounded degree networks with high routing requirements. To illustrate our technique we first give an unbounded degree variant of the network constructed in section 3. Later we extend our construction to bounded degree networks. From now on and for the rest of this paper we assume that $P = (p_{i,j})$ is a Kolmogorov random $p \times q$ matrix satisfying

$$K(P) = pq + O(\log(pq)), \tag{2}$$

where $p, q \in \Omega(\log(pq))$.

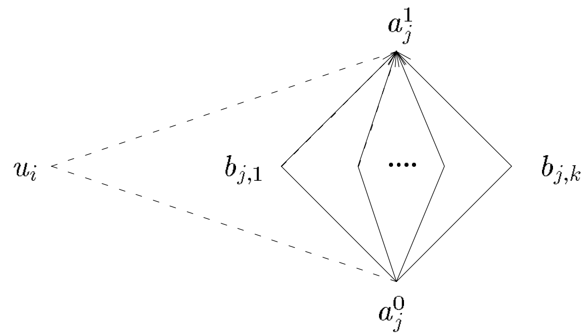

Figure 2: The (i,j)th component of the network $G_{P,k}$.

The network we construct, denoted by $G_{P,k}$, is depicted in Figure 2 and consists of the following.

1. A collection of $p+2q+kq$ vertices consisting of
   - an independent set $u_1, u_2, \ldots, u_p$,
   - $a_1^b, a_2^b, \ldots, a_q^b$, where $b = 0, 1$, and
   - $b_{j,1}, b_{j,2}, \ldots, b_{j,k}$, $j = 1, 2, \ldots, q$.
2. A collection of q copies of the following graph; the jth copy consists of the vertices $a_j^0, a_j^1, b_{j,1}, b_{j,2}, \ldots, b_{j,k}$ and the edges

$$\{a_j^0, b_{j,1}\}, \{a_j^0, b_{j,2}\}, \ldots, \{a_j^0, b_{j,k}\},$$

$$e_{j,r} := (b_{j,r}, a_j^1), r = 1, 2, \ldots, k$$

   (note that the last k edges above are assumed to be directed).
3. For each $i = 1, 2, \ldots, p, j = 1, 2, \ldots, q$ and $b = 0, 1$,

$$\{u_i, a_j^b\} \text{ is an edge } \Leftrightarrow p_{i,j} = b.$$

Clearly, the network $G_{P,k}$ has $p + (k+2)q$ vertices, $\Theta((p+2k)q)$ edges (this follows from [12][Theorem 2.15, page 131], and maximum degree $\max\{q, k+p\}$. Like Theorem 6 we can prove the following result.

THEOREM **11** *$G_{P,k}$ is a connected network of diameter 4.* ■

The main theorem is the following.

THEOREM **12** *Consider the network $G_{P,k}$, which has $p+(k+2)q$ vertices and $\Theta((p+2k)q)$ edges. Then for every near optimal routing scheme with stretch factor $s<2$ the total number of memory bits is bounded below by $\Omega(kpq)$.*

PROOF (OUTLINE) Similar to the proofs of Theorems 7 and 9 using the fact that P is a matrix of s-constraints for the graph $G_{P,k}$, for any $s<2$ and $k\geq 1$. One need only observe that we can prove the lower bound for k identical copies of routers. The constants in $p,q\in\Omega(\log(pq))$ depend on the definition of incompressibility [12]. ■

If $p=k=\Theta(n/\log n)$ and $q=\Theta(\log n)$ then we obtain a network having maximum degree $p=\Theta(n/\log n)$, and $\Theta(n)$ edges which requires $\Omega(n^2/\log n)$ memory bits for any near optimal routing scheme with stretch factor <2.

4.1 Bounding the degree

Next we prove lower bound results for bounded degree networks. The basic network (depicted in Figure 3), denoted by $G_P[G_1,\ldots,G_q]$, is based on the construction of Gavoille and Guévremont [9, 10] and is defined from a Kolmogorov random $p\times q$ matrix $P=(p_{i,j})$ (satisfying (2)) and q subgraphs $G_1,\ldots,G_q$. The graph $G_P[G_1,\ldots,G_q]$ consists of the following.

1. Vertices $w_{i,j},w^a_{i,j},w^b_{i,j}$ for $i=1,\ldots,p$ and $j=1,\ldots,q$.
2. Vertices $v_1,v_2,\ldots,v_p$. For each i vertex v_i is the root of a tree $T(v_i)$ of degree $\leq d$ with leaves the vertices $w_{i,j}$, $j=1,\ldots,q$. The trees $T(v_i)$ are isomorphic and vertex and edge disjoint.
3. Each graph G_j has degree $\leq d$ and two distinguished edges $A_j=\{a_j,\bar{a}_j\}$ and $B_j=\{b_j,\bar{b}_j\}$. Vertex a_j (respectively, b_j) is the root of a tree $T'(a_j)$ (respectively, $T'(b_j)$) of degree $\leq d-1$ with leaves the vertices $w^a_{i,j}$ (respectively, $w^b_{i,j}$), $i=1,\ldots,p$. The trees are isomorphic and vertex and edge disjoint.
4. In addition[†] we assume that

$$\{w_{i,j},w^a_{i,j}\}\text{ is an edge }\Leftrightarrow p_{i,j}=1,$$

[†]It will be convenient to think of the vertices $w_{i,j},w^a_{i,j},w^b_{i,j}$ as lying on the (i,j)th quadrant of a $q\times p$ mesh determined by the ith row and jth column (see Figure 4).

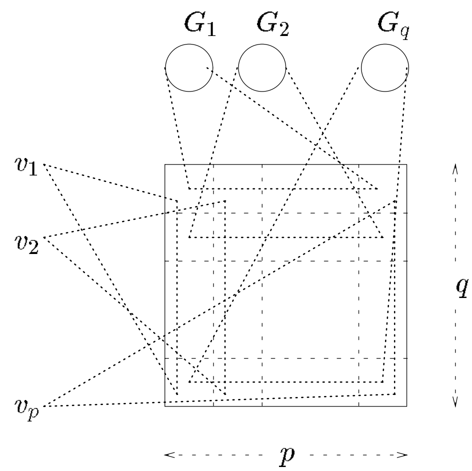

Figure 3: The network $G_P[G_1, \ldots, G_q]$.

$$\{w_{i,j}, w^b_{i,j}\} \text{ is an edge } \Leftrightarrow p_{i,j} = 0.$$

In the sequel we define the graphs G_j which are appropriate for proving strong lower bounds for interval and near optimal routing.

4.2 Interval routing

The graph G_j is depicted in Figure 5. It consists of two isomorphic trees each of degree $\leq d-1$ rooted at the vertices $\bar{a}_j$ and $\bar{b}_j$, respectively, and k leaves each, say,

$$l_1(a_j), l_2(a_j), \ldots, l_k(a_j), l_1(b_j), l_2(b_j), \ldots, l_k(b_j).$$

Leaves are joined by the directed edges $e_{j,r} := (l_r(b_j), l_r(a_j))$, $r = 1, 2, \ldots, k$. As with Theorems 6, 11 we can prove the following result.

THEOREM **13** *$G_P[G_1, \ldots, G_q]$ is a connected network of degree $\leq d$ and diameter $O(\log_{d-1} k + \log_{d-1} p + \log_d q)$.* ∎

It is easy to see that the matrix of constraints having $v_1, \ldots, v_p$ as row-indices and the directed edges $(l_r(b_j), l_r(a_j)), j = 1, 2, \ldots, q$, as column-indices, for each fixed $r \leq k$, is identical to the matrix P. Thus we obtain k copies of the matrix of constraints and we have the following theorem.

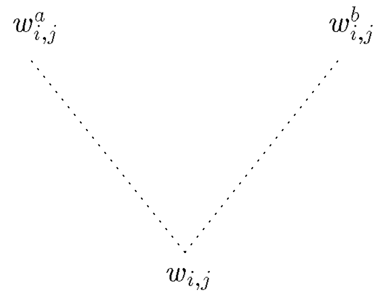

Figure 4: The links of the (i,j)th quadrant of the network $G_P[G_1,\ldots,G_q]$ connecting vertices $w^a_{i,j}, w^b_{i,j}, w_{i,j}$.

THEOREM **14** *$G_P[G_1,\ldots,G_q]$ has $O(pq+kq)$ vertices and the same number of edges, and maximal degree $\leq d$. Moreover, for every interval routing scheme there exist $\Omega(kq)$ routers each requiring $\Omega(p)$ intervals at some link.* ■

In particular, if we choose $p = k = \Theta(n/\log n)$ and $q = \Theta(\log n)$ then the resulting network has $O(n)$ vertices, the same number of edges, maximal degree $\leq d$, and $\Omega(n)$ routers each requiring $\Omega(n/\log n)$ intervals on some link of each of these routers for every interval routing scheme.

4.3 Optimal universal routing

The network of subsection 4.2 can also be used to analyze universal routing with stretch factor $s = 1$.

THEOREM **15** *$G_P[G_1,\ldots,G_q]$ has $O(pq+kq)$ vertices and the same number of edges, and maximal degree $\leq d$. Then every optimal routing scheme requires a total of at least $\Omega(kpq)$ memory bits.* ■

In particular, if we choose $p = k = \Theta(n/\log n)$ and $q = \Theta(\log n)$ then the resulting network has $O(n)$ vertices, the same number of edges, maximal degree $\leq d$, and requires a total of at least $\Omega(n^2/\log n)$ memory bits, for any optimal routing scheme.

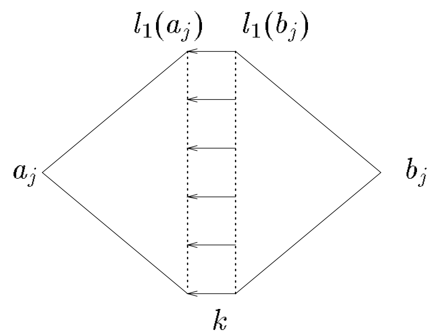

Figure 5: The (i, j)th component of the graph G_j used for the lower bounds on interval routing.

4.4 Near optimal universal routing

The construction in this case is based on a modification of the graphs $G_P[G_1, \ldots, G_q]$. We transform the trees $T'(a_j), T'(b_j)$ by replacing each of their leaves with a chain of length b.[‡] Let $\bar{T}'(a_j), \bar{T}'(b_j)$ be the resulting trees.

The graph G_j is depicted in Figure 6. It consists of two isomorphic trees of degree $\leq d-1$ rooted at the vertices $\bar{a}_j$ and $\bar{b}_j$, respectively, each having k leaves, say, $l_1(a_j), l_2(a_j), \ldots, l_k(a_j)$, $l_1(b_j), l_2(b_j), \ldots, l_k(b_j)$. We join the leaves with chains $C_{r,j} := C(l_r(b_j), l_r(a_j)), r = 1, 2, \ldots, k$ each of length $2b+1$. We denote the resulting network by $\bar{G}_P[G_1, \ldots, G_q]$. Let e_j be the middle edge of this chain.

Unlike the case of interval routing we use the matrix of s-constraints, $s < 2$, in the following slightly extended form. It will follow from the argument of Theorem 17 that for fixed $r = 1, 2, \ldots, k$, from the routing information of all the processors in all the chains $C_{r,j}, j = 1, 2, \ldots, q$ we can reconstruct the matrix P. Thus the total number of memory bits required on all the processors of all these chains is at least $\Omega(kpq)$.

As with Theorems 6, 11, 13 we can prove the following result.

THEOREM **16** *$\bar{G}_P[G_1, \ldots, G_q]$ is a connected network of degree $\leq d$ and diameter $2b + O(\log_{d-1} k + \log_{d-1} p + \log_d q)$.* ■

[‡]It will follow from the proof of Theorem 17 that the value of the parameter b can be chosen to be $O(\log n)$.

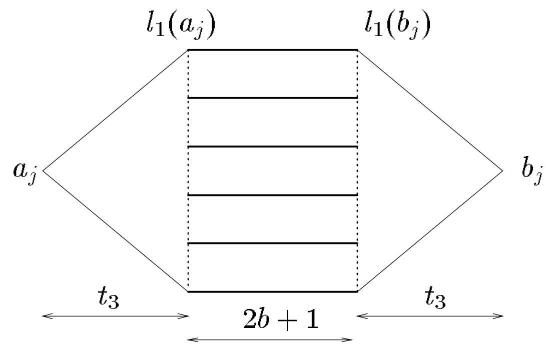

Figure 6: The graph G_j used for the lower bounds on near optimal routing.

The main theorem is the following.

THEOREM **17** *$\bar{G}_P[G_1, \ldots, G_q]$ has $O(pq+kqb)$ vertices and the same number of edges, and maximal degree $\leq d$. Moreover, for every near optimal routing scheme with stretch factor $s < 2$ the total number of memory bits required by all the routers on all the chains $C_{r,j}, r = 1, 2, \ldots, k, j = 1, 2, \ldots, q$ is bounded below by $\Omega(kpq)$.*

PROOF (OUTLINE) Assume that $\mathcal{RF}$ is a routing function for near optimal shortest paths with stretch factor s sufficiently close to 2, but $s < 2$. Consider routing from the tail of e_j to v_i. Define $t_1 = \log_d p, t_2 = \log_d q, t_3 = \log_{d-1} k$. Notice that the trees $\bar{T}'(a_j), \bar{T}'(b_j)$ have height $t_2 + b$. Observe that regardless of the value of $p_{i,j}$ the shortest path from the tail of e_j to v_i has length

$$2b + t_1 + t_2 + t_3 + \Theta(1).$$

A routing function either uses link $\{w_{i,j}, w^a_{i,j}\}$ when it should use $\{w_{i,j}, w^b_{i,j}\}$ or vice versa.

Let $L_{r,j}$ and $R_{r,j}$ be the left- and right-most endpoints of the chain $C_{r,j}$, respectively. We reconstruct the matrix P of constraints from the given routing function $\mathcal{RF}$ by showing that

- $p_{i,j} = 1 \Leftrightarrow$ "every walk with stretch factor s from the tail of e_j to v_i must exit from vertex $L_{r,j}$", and
- $p_{i,j} = 0 \Leftrightarrow$ "every walk with stretch factor s from the tail of e_j to v_i must exit from vertex $R_{r,j}$".

This follows from the argument below.
Case 1: Assume that $p_{i,j} = 1$.
Now consider a walk from the tail of e_j to v_i that does not exit from $L_{r,j}$. We have two cases.
Subcase 1a: walk uses vertex a_j. It is easy to see that the resulting path will have length at least $4b + t_1 + t_2 + t_3 + \Theta(1)$.
Subcase 1b: walk uses vertex b_j. As before, it is easy to see that the resulting walk will have length at least $4b + t_1 + t_2 + t_3 + \Theta(1)$.
Case 2: Assume that $p_{i,j} = 0$.
Now consider a walk from the tail of e_j to v_i that does not exit from $R_{r,j}$. We have two cases.
Subcase 2a: walk uses vertex a_j. It is easy to see that the resulting path will have length at least $4b + t_1 + t_2 + t_3 + \Theta(1)$.
Subcase 2b: walk uses vertex b_j. As before, it is easy to see that the resulting path will have length at least $4b + t_1 + t_2 + t_3 + \Theta(1)$.

Combining the above inequalities we see that the resulting stretch factor is at least

$$\frac{4b + t_1 + t_2 + t_3 + \Theta(1)}{2b + t_1 + t_2 + t_3 + \Theta(1)}$$

Now choose $b = c(t_1 + t_2 + t_3)$ and we obtain that the ratio of the resulting path over the optimal path has stretch at least $\frac{4c+1}{2c+1}$ Now the theorem follows from the fact that the above number is arbitrarily close to 2 as c approaches sufficiently close to ∞. The theorem now follows from Lemma 5. ■

In particular, if we choose $p = \Theta(n/\log n), k = \Theta(n/\log^2 n)$ and $q = b = \Theta(\log n)$ then the resulting network has maximal degree $\leq d$ and $\Omega(n/\log n)$ routers each requiring $\Omega(n/\log n)$ bits at some link on every near optimal routing scheme with stretch factor < 2.

Conclusion

The main contribution of the present paper is to obtain the best known worst-case bounds on interval routing on bounded and unbounded degree networks. Our technique was also used to obtain bounds on near optimal routing. The Kolmogorov complexity analysis provided is generally applicable in a straightforward manner to other networks by randomizing over an appropriate subset of their edge-set. Fraigniaud and Gavoille in [5] also give an optimal lower bound on local storage by constructing a network which has a single router requiring $\Omega(n \log n)$ memory bits, for any near optimal routing scheme with stretch factor < 2. Our Kolmogorov complexity analysis is also appplicable here. For any $d \in \Omega(\log n), d \leq n$ we can show the existence of graphs which have $\Omega(n/d)$ routers each requiring $\Omega(n \log d)$ bits of local memory. However, the general problem on whether $\Omega(n^2 \log n)$ is a lower bound on universal routing still remains open.

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