On Tilable Orthogonal Polygons*

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Abstract

We consider rectangular tilings of orthogonal polygons with vertices located at integer lattice points. Let G be a set of reals closed under the usual addition operation. A G-rectangle is a rectangle at least one of whose sides is in G. We show that if an orthogonal polygon without holes can be tiled with G-rectangles then one of the sides of the polygon must be in G. As a special case this solves the conjecture that domino tilable orthogonal polygons must have at least one side of even length. We also explore separately the case of othogonal polygons placed in a chessboard. We establish a condition which determines the number of black minus white squares of the chessboard occupied by the polygon. This number depends exclusively on the parity sequence of the lengths of the sides of the orthogonal polygon. This approach produces a different proof of the conjecture of the non domino-tilability of of orthogonal polygons without even length sides. We also give some generalizations for polygons with holes and polytopes in 3 dimensions.

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1 Introduction

In this paper we are concerned only with simple orthogonal polygons with vertices located at lattice points in the plane. We study the impact that the existence of certain cuts of the orhogonal polygon into rectangles have on the length of the sides of the polygon.

A problem related to our study is one that is a generalization of a problem previously posed by A. Shen in the Mathematical Intelligencer [5]:

Problem 1.1 *A simple orthogonal polygon is cut into sevaral semi-integer (i.e., either the vertical or horizontal side is an integer) rectangles (whose sides are rectilinear). Does the polygon have a side which is an integer?*

Our study answers Problem 1.1 and provides a general result (Theorem2.1) concerning simple orthogonal polygons without holes which can be cut by G-rectangles (these are rectangles for which either the horizontal or vertical side has length in the set G).

Our analysis looks in detail to the case of *domino* tilings: this is an important concept used throughout the paper.

Definition 1.1 A *domino* is a rectangle formed by two unit squares adjacent along an edge.

It is well-known that a rectangle of integer side-lengths can be tiled by dominoes if and only if at least one side of the rectangle is of even length [2]. The proof of this is based on the simple observation that a region tiled by dominoes must always have an even number of unit squares. Therefore if the tilable region is a rectangle then one of its sides must be even.

There are several interesting ways to look for generalizations of this result: either by generalizing rectangle to simple polygon or domino to polyomino or both. This gives rise to the following problem.

Problem 1.2 *If a simple orthogonal polygon can be tiled by dominoes does it have a side of even length?*

Along the same lines, an interesting conjecture arises from a generalization of de Bruijn's well-known theorem: if a rectangle is tilable by polyominoes with sides $1 \times p$ then one of its sides must have length which is a multiple of p (see [1, 6]).

Problem 1.3 *If a simple orthogonal polygon can be tiled by $1 \times p$ polyominoes, does it have a side of length a multiple of p?*

In this paper we answer both Problems 1.2 and 1.3. In particular, we prove that the above stated theorem is true in a much more general form, for all orthongonal simple polygons without holes. (If the polygon has a hole then this may not be true, as depicted by the left picture in Figure 1.) A complication arises because an orthogonal polygon (as opposed to a rectangle) of odd sides may not be possible to tile although it has an even number of unit cells (see Figure 1).

Figure 1: The polygon to the left has a hole (dashed square), it is tilable by dominoes and all its sides have odd length. The polygon to the right is a non-tilable orthogonal polygon with the same number of black and white unit-squares.

The technique which is described in Section 2 is applicable to both Problem 1.2 and Problem 1.3. Section 3 focuses only to domino tilings of orthogonal polygons. The polygon sides are drawn along chessboard lines in which case some black and white squares of the chessboard fall within the interior of the polygon. A fundamental observation in our proof of the main theorem for domino tilings (see Theorem 3.1) is based on the fact that a chessboard coloring of an orthogonal polygon of odd sides cannot have the same number of black and white squares. Thus, it cannot be tiled by dominoes, since each domino occupies one black and one white square. This raises the question if all polygons containing the same number of black and white cells can be tiled by dominoes. The answer is *no*, as the polygon on Figure 1 shows.

Section 3 characterizes the relation between on the one hand the difference of black and white squares covered by the polygon and on the other the parity sequence of the sequence of lengths of the sides of the polygon. As a corollary of this relation we can observe that an orthogonal polygon of odd side lengths covers a different number of black and white squares. This gives a different proof to the answer of Problem 1.3.

The following definition provides useful terminology that will be used in the sequel.

Definition 1.2 Let P be an orthogonal polygon of integer side lengths. Let R be the smallest rectangle of vertical and horizontal sides containing P. Suppose R is an $n \times m$ rectangle. Divide R into nm unit squares, and denote by $T_{i,j}(P)$ (or simply $T_{i,j}$) the unit square in the i-th row and j-th column. Of course, some of these cells are in polygon P, others are not. A *chessboard coloring* of an orthogonal polygon is a coloring of its unit squares $T_{i,j}$ by black and white, such that any two squares which share a side are colored with different colors.

2 A General Tiling Theorem

In this section we consider general rectangular tiles.

Definition 2.1 *Let G be a nonempty set. G is called additive if for any x, y in G their sum $x + y$ is also in G.*

Here $+$ is a binary operation on G and may stand either for the usual sum among real numbers, or even addition modulo m for some integer m. Typical examples of additive sets

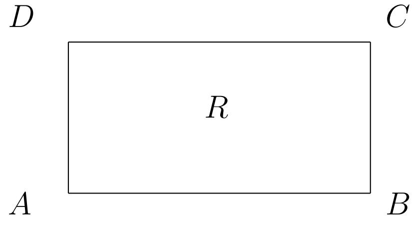

Figure 2: A G-rectangle R with vertices A, B, C, D.

G of reals include the set of real numbers, the set of rational numbers, the set of integers, or for some integer m the set of integer multiples of m. Another example is addition mod$m$. Thus, if $m = 2$ and addition is mod2 then a G-rectangle is a domino [2].

Definition 2.2 *For an additive set G of reals we call G-rectangle (or G-tile) a rectangle such that the length of either its vertical or its horizontal side is in G. A G-tiling of an orthogonal polygon P is a tiling for which the tiles are G-rectangles. An otthogonal polygon is G-tilable if it can be tiled by G-tiles.*

The main theorem of this section is the following.

Theorem 2.1 *Let G be an additive set. If a simple orthogonal polygon P without holes is G-tilable then at least one of its sides must be in G.*

Proof A G-rectangle is determined by the four endpoints A, B, C, D (see Figure 2). To any G-rectangle we associate two horizontal and two vertical "edges":

$$\{A, B\}, \{D, C\} \text{ and } \{A, D\}, \{B, C\},$$

respectively. If the width (respectively, height) of the rectangle is in G then both of the horizontal edges (respectively, both of the vertical) edges, are in the graph; if both horizontal and vertical lengths are in G then we select either the two horizontal or the two vertical edges but not both. Consider a G-tiling of the orthogonal polygon. This G-tiling gives rise to a graph on the set of vertices of the G-tiles (which also includes all the vertices of the original poygon) that may be disconnected. The following properties of this graph are easy to prove from the definitions.

1. The graph is planar.

2. The original vertices of the polygon have odd degree.

3. All other vertices have even degree.

We can prove the following claim.
Claim. There are two vertices on the same side of the orthogonal polygon which are connected by a path of the graph.
Proof of the Claim. Indeed, take two vertices, say u, v, of the polygon which are connected by a path of the graph such that the number of sides of the polygon between

these two vertices is minimal. Assume on the contrary that u and v do not lie on the same side of the polygon. In this case either there are two other vertices of the polygon "between" u, v which are connected by a path or else there is a single vrtex w of the polygon "between" u and v and a path emanating from w to another vertex w' which by planarity crosses the path between u and v. However, in both cases this contradicts the minimality of the path between u and v. This completes the proof of the claim.

Returning to the proof of the theorem we argue as follows. Take two vertices u and v of the polygon which lie on the same side, say e, of the polygon and are also connected by a path in the graph. Using the additivity of the set G it is easy to see that the length of e is in G. This completes the proof of the theorem. ■

The following corollaries are immediate from the theorem. Let Q, Z denote the sets of rational, integer numbers, respectively.

Corollary 2.1 *If a simple orthogonal polygon is tilable by Q-rectangles then one of its sides must have length which is a rational.* ■

Corollary 2.2 *If a simple orthogonal polygon is tilable by Z-rectangles then one of its sides must have length which is an integer.* ■

Corollary 2.3 *If for some integer p, a simple orthogonal polygon is tilable by $1 \times p$ or $p \times 1$ rectangles then one of its sides must have length which is a multiple of p.* ■

Corollary 2.4 *If a simple orthogonal polygon is tilable by dominoes then one of its sides must have even length.* ■

3 Orthogonal Polygons Placed in a Chessboard

In this section we will suppose that the integer lattice containing the polygon is a chessboard. The polygon sides traverse edges of the squares of the chessboard and every such square falls entirely either inside or outside the polygon. We can ask what is the number of white and black squares of the chessboard being inside the polygon. If the number of such white squares is not equal to the number of black squares then we will be able to conclude that the polygon is not tilable by dominoes. We will prove the following theorem.

Theorem 3.1 *Let P be an orthogonal polygon without holes such that all sides are of odd length. Then P cannot be tiled by dominoes.*

PROOF . Take a chessboard coloring of P. We are going to prove that the number of black and white cells are not the same. This implies the theorem, since every domino occupies one black and one white square.

We call $T_{i,j}$ a *corner square* if $T_{i,j} \subset P$ and at least two adjacent sides of the square fall onto sides of P. We may suppose that one of the corner squares of P is black (otherwise we can reverse the coloring). We will use the following two lemmas, which we prove later.

Lemma 3.1 *If one of the corner squares of P is black, then every corner square is black.*

Lemma 3.2 *Suppose P has more than one row, and all corner squares of P are black. Then one can cut off n cells $C = \{T_1, \ldots, T_n\}$ ($n > 0$) from the polygon P, so that the following conditions are satisfied:*

(i) $P - C$ is a set of orthogonal polygons $P_1^*, \ldots, P_k^*$, all of which have odd side lengths ($k \geq 1$) and all corner squares of all of them are black.

(ii) If there is only one remainder polygon ($k = 1$), then C contains at least as many black squares as white ones.

(iii) If there are more than one remainder polygons ($k \geq 2$), then C contains at most one more white square than black one.

It is easy to see that Lemma 3.2 implies Theorem 3.1. Indeed, we can keep recursively cutting off cells from the remainder polygon or polygons. In the end polygons with at most one row remain, where the length of the row is always odd. Therefore the excess number of black squares is the same as the number of polygons. In the process, when the number of polygons increases, we cut off at most one more white cell than black one, thus the original polygon contained more black cells than white ones. This completes the proof of Theorem 3.1 assuming the two lemmas. ∎

Now all that is left is the proof of the two lemmas.

Proof of Lemma 3.1. Let us divide each edge of the polygon into unit segments. Color the segments by the color of the squares (in P) those segments are the sides of.

First, observe that each edge has its first and last segment colored by the same color, because the consecutive segments have alternating colors, and the length of the edge is odd. Also, any two segments joining at a corner (positive or negative) must have the same color. It follows that since at least one edge has its first segment colored black, all edges have their first and last segments colored black, therefore all internal corner squares are black. This completes the proof of Lemma 3.1. ∎

As before, one can observe (using similar arguments) that all external corner squares are white. By external corner square we mean a square not in P but adjacent to two perpendicular edges of P.

Proof of Lemma 3.2. Suppose first that the polygon contains a square ($T_{x,y}$), which is bounded on 3 sides by the boundary of P. Without loss of generality, we may assume that the only neighbor of $T_{x,y}$ inside the polygon P is $T_{x+1,y}$. It is clear that $T_{x,y}$ is black, while $T_{x+1,y}$ is white. It is easy to see that there are only 3 distinct cases (up to rotation and symmetry) depending upon what neighbors of $T_{x+1,y}$ are inside the polygon (see Figure 3 (a), (b) and (c)).

In cases (a) and (b) (see Figure 3) we may take the cut $C = \{T_{x,y}, T_{x+1,y}\}$, which satisfies the conditions of the lemma. Indeed, in both cases some sidelengths of P are decreased by 2 (thus they remain odd) and all new sides are of length 1. Also, all new corner squares of P are black.

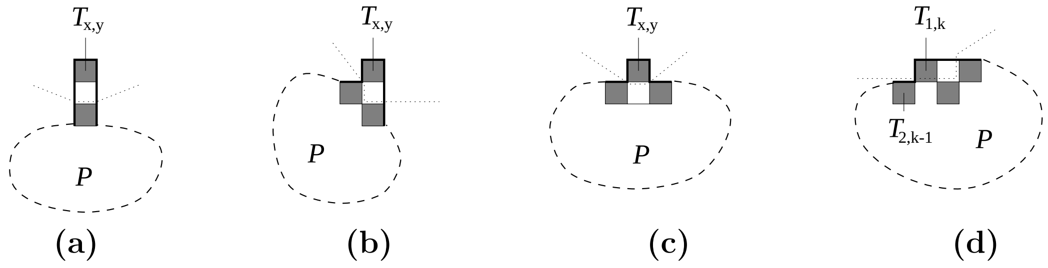

Figure 3: The four cases considered in the proof of Lemma 3.2.

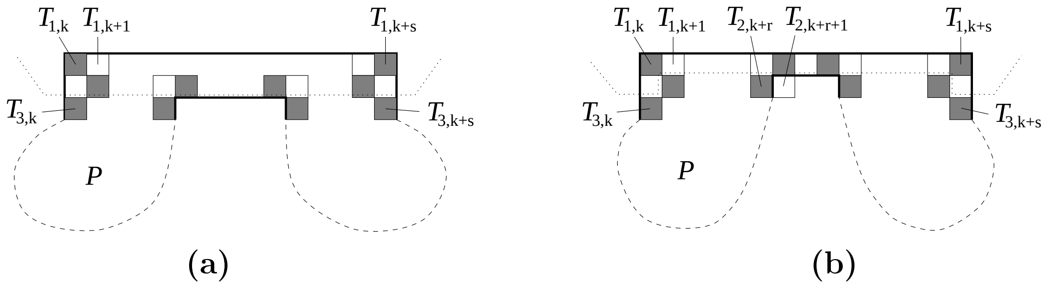

Figure 4: Two subcases in the proof of Lemma 3.2.

In case (c) we simply cut off $C = \{T_{x,y}\}$ (see Figure 3(c)). The newly created edge of P along the bottom of $T_{x,y}$ will have odd length, since it's length is the sum of two previous edge lengths plus 1. No new corner squares have been created.

Therefore we may suppose that all squares in the polygon have at least two neighbors inside P. Let $T_{1,k}$ be the leftmost square of the first row which is inside P (so $T_{1,k-1} \not\subset P$). Let s be the number for which

$$\{T_{1,k}, T_{1,k+1}, \ldots, T_{1,k+s}\} \subset P \qquad \text{but} \qquad T_{1,k+s+1} \not\subset P.$$

Clearly, $s \geq 2$, s is an even number, and $T_{1,k+s}$ is black. We also know that $T_{2,k} \subset P$.

Suppose first that $T_{2,k-1} \subset P$. The proof of Lemma 3.1 implies that all external corner squares are white, therefore $T_{2,k+1} \subset P$ (see Figure 3(d)). Then we can take the cut $C = \{T_{1,k}, T_{1,k+1}\}$. This shortens an edge by 2 and lengthens another by 2. The two potential new corner squares are $T_{1,k+2}$ and $T_{2,k+1}$, both black. Thus C is a good choice.

The only case left is when $T_{2,k-1} \not\subset P$. We can similarly suppose that $T_{2,k+s+1} \not\subset P$. The fact that all edges are of odd length enforce that $T_{3,k} \subset P$ and $T_{3,k-1} \not\subset P$. Again we similarly conclude that $T_{3,k+s} \subset P$ and $T_{3,k+s+1} \not\subset P$. Thus, we have (see Figure 4)

$$\{T_{2,k}, T_{3,k}, T_{2,k+s}, T_{3,k+s}\} \subset P, \text{ and } \{T_{2,k-1}, T_{3,k-1}, T_{2,k+s+1}, T_{3,k+s+1}\} \cap P = \emptyset.$$

We distinguish 2 subcases, depending on what elements of the second row are in P.

(i) $\{T_{2,k}, \ldots, T_{2,k+s}\} \subset P$.

Take $C = \{T_{1,k}, \ldots, T_{1,k+s}, T_{2,k}, \ldots, T_{2,k+s}\}$ (see Figure 4(a)). This set contains an equal number of black and white squares, and by cutting off C, we decrease each of two side-lengths by 2. To see that all corner squares remain black, note the following. At least two new corner squares (besides $T_{3,k}$ and $T_{3,k+s}$) are created if $T_{3,i} \not\subset P$ for some $k < i < k + s$. But, since all first and last unit segments of each edge has been colored black, it follows that each square inside P and next to an edge in line 3 must be black. But those are exactly the squares that become corner squares. The fact that all new corner squares are black implies that all new edge lengths are odd.

(ii) $\{T_{2,k}, \ldots, T_{2,k+r}\} \subset P$, but $T_{2,k+r+1} \not\subset P$ for some r ($r < s$) (see Fig. 3(b)).

Now take $C = \{T_{1,k}, \ldots, T_{1,k+s}, T_{2,k}, T_{2,k+s}\}$. In this case there are one more white squares in C than black ones, but $T_{2,k+r+1} \not\subset P$ guarantees that the polygon falls into two parts after the cut. Following a similar reasoning as for case (i), we can again conclude that all new corner squares are black, in particular $T_{2,k+r}$ must be black, therefore the new side lengths are all odd. Thus, C satisfies the conditions of the lemma.

This concludes the proof of Lemma 3.2. ■

3.1 Black versus white squares

We are now in a position to characterize the difference between the number of black and the number of white squares in an orthogonal polygon all of whose sides are odd. First we need to prove a simple lemma.

Lemma 3.3 *Every orthogonal polygon with sides of odd length has a total of* $4n$ *sides for some integer* n.

Proof . Let us define a coordinate system such that the origin corresponds to one of the vertices of the polygon. If we go around the boundary of the polygon in clockwise order, jumping from one vertex to the next, then the horizontal and vertical jumps are alternating, so it is enough to prove that the number of horizontal jumps is even. But after every horizontal jump, the parity of the x-coordinate of the vertex will change, while during vertical jumps it remains the same. So since the start and goal coordinates are 0, we must make an even number of horizontal jumps. ■

The following definition will be important in the sequel.

Definition 3.1 Let $N_{b-w}(P)$ denote the number of black squares minus the number of white squares covered by a polygon P. Let $S(P)$ denote the number of sides of P.

Theorem 3.2 *In an orthogonal polygon* P *of odd side lengths* $S(P) = 4 \cdot |N_{b-w}(P)|$. *In other words, if* P *has* $4n$ *sides, the difference of the number of black and white squares in* P *is* n.

PROOF . Without loss of generality we may suppose that $N_{b-w}(P) \geq 0$. Observe, that if an orthogonal polygon P has only one row, then it has 4 sides, so $S(P) = 4$ and $N_{b-w}(P) = 1$. Thus, the theorem is true for all polygons with only one row. Therefore, by using induction, it is enough to prove that if we make a cut satisfying the conditions of Lemma 3.2, then $\sum_i S(P_i^*) = 4 \cdot \sum_i N_{b-w}(P_i^*)$ implies that $S(P) = 4 \cdot N_{b-w}(P)$.

There are 6 different cuts described in the proof of Lemma of 3.2. For the cuts on Figures 3 (a), (b) and (d) $\sum_i N_{b-w}(P_i^*) = N_{b-w}(P)$ and also $\sum_i S(P_i^*) = S(P)$. For the cut on Fig. 2(c) $\sum_i N_{b-w}(P_i^*) = N_{b-w}(P) - 1$, but $\sum_i S(P_i^*) = S(P) - 4$, so those cuts all satisfy the conditions.

For the cut on Figure 4(a), $\sum_i N_{b-w}(P_i^*) = N_{b-w}(P)$ and $\sum_i S(P_i^*) = S(P)$, because if originally there were m distinct horizontal edges between the second and third rows, then all those disappear in addition to the upmost edge, while $m+1$ new edges are created between the second and third rows.

Finally, for the cut on Figure 4(b) $\sum_i N_{b-w}(P_i^*) = N_{b-w}(P) + 1$ and $\sum_i S(P_i^*) = S(P) + 4$, this latter because in place of the uppermost edge and the m edges between the first and second row there are $5 + m$ new edges.

This proves that all cuts preserved the equality $S(P) = 4 \cdot N_{b-w}(P)$, therefore the original polygon must satisfy the same equality. ■

3.2 Polygons with odd and even sides

In this section we prove that for a polygon with arbitrary integer edge lengths $N_{b-w}(P)$ depends only on the parities of the sequence of its edges.

Definition 3.2 Let P denote any orthogonal polygon with n sides. Let us follow the edges of P in a clockwise order (starting at an arbitrary position) and write down a letter 'e' whenever we encounter an edge of even length, and write an 'o' for an odd edge. Let us call such a sequence of 'e's and 'o's the *parity sequence* of polygon P. Two sequences are considered equivalent, if they can be derived from the same polygon by choosing a different starting position.

Similarly, follow the edges of P in clockwise order, and write down a letter 'c', when there is a clockwise turn at a vertex, and write an 'r' for each reverse (counter-clockwise) turn. Let us call the resulting sequence the *turn sequence* of polygon P.

As an example, the polygon on Figure 1 has parity sequence

'ooooeooooeo'='eoooooeooooo'=...

and turn sequence

'crccrccrccrc'='rccrccrccrcc'=... .

Observe that every turn sequence contains 4 more 'c' than 'r'. Also, every polygon with more than 4 edges has at least one 'r' in its turn sequence.

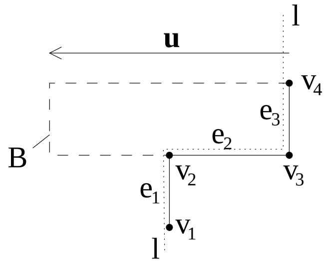

Figure 5: Four consecutive edges of a polygon.

Theorem 3.3 *Every orthogonal polygon P with $N_{b-w}(P) \neq 0$ (with odd and/or even sides) can be transformed into an orthogonal polygon P^* with odd side lengths only, where $N_{b-w}(P^*) = N_{b-w}(P)$.*

For the proof of Theorem 3.3 we will need the following lemma.

Lemma 3.4 *Suppose that e_1, e_2, e_3 and e_4 are four consecutive edges of a polygon P (in clockwise order) and the vertices enclosed by the above edges are v_2, v_3 and v_4 (see Figure 5.). Suppose that v_2 and v_3 are represented by 'cr' in the turn sequence. Then we can transform P into P' such that*

(a) The only difference in the turn sequences of P and P' is that in P' v_2 and v_3 are represented by 'rc' instead of 'cr'.

(b) The parity sequences of P' and P are the same.

(c) $N_{b-w}(P') = N_{b-w}(P)$.

Proof of Lemma 3.4. Suppose that the coordinates of v_2 and v_3 are (x_2, y_2) and (x_3, y_3) respectively. Without loss of generality we may suppose that e_1 is a vertical edge. Define the broken line l as the line connecting the points $(x_3 - \epsilon, \infty), (x_3 - \epsilon, y_3 + \epsilon), (x_2 - \epsilon, y_2 + \epsilon), (x_2 - \epsilon, -\infty)$, where ϵ is a small number (see Figure 5). Clearly, l doesn't go through any vertices of P, and it intersects only horizontal edges.

Get polygon P_1 from P by shifting all vertices of P to the left of l by the vector $\mathbf{u} = 2\overrightarrow{v_3v_2}$. The length of the vertical edges of P will not change, while those horizontal edges, which were intersected by l will be extended by $|\mathbf{u}|$, an even number. Thus, the turn sequences and parity sequences of P and P_1 are the same. Also, since every row of P is either unchanged, or had been extended by $|\mathbf{u}|$, an even number, we conclude that $N_{b-w}(P_1) = N_{b-w}(P)$.

As a result of the above transformation, there is no vertex now in the box B with lower right corner at v_3 and with horizontal edge length $|\mathbf{u}|$ and vertical edge length $|\overrightarrow{v_3v_4}|$. On the boundary of B, the only vertices are v_2, v_3 and v_4.

Now get P' from P_1 by shifting v_3 and v_4 by $\mathbf{u}$. This transformation doesn't change the length of any edges except for e_4. If v_4 was a 'c' turn, then the length of e_4 is increased by $|\mathbf{u}|$, if it was an 'r' turn, then the length of e_4 is decreased by $|\mathbf{u}|$. Note, that in the

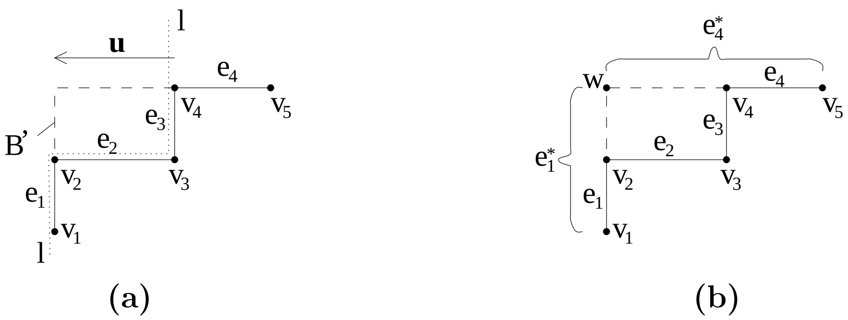

Figure 6: Transforming the turn sequence of the polygon.

latter case e_4 had just been increased by $|\mathbf{u}|$ in the previous step, so it is always possible to decrease its length by $|\mathbf{u}|$.

The parity sequences of P_1 and P' are the same. However v_2 is changed into an 'r' turn, while v_3 is changed into a 'c' turn. The volume of P_1 is changed by the squares in box B, but since there is an equal number of black and white squares in this box, $N_{b-w}(P_1) = N_{b-w}(P')$. This proves Lemma 3.4. ■

Proof of Theorem 3.3. As any polygon P with four sides and $N_{b-w}(P) \neq 0$ must have all sides of odd length, so we may suppose that P has more than four sides. We are going to show an algorithm for transforming P into P^*, a polygon with odd sides, whenever $N_{b-w}(P) \neq 0$. Without loss of generality we suppose that $N_{b-w}(P) > 0$.

Lemma 3.4 implies that any 'cr' in the turn sequence can be transformed into an 'rc', and similarly any 'rc' can be transformed into a 'cr'. Since we have at least one 'r' and at least five 'c's in the turn sequence, any three consecutive positions in the turn sequence can be transformed into 'crc'.

First we show that any polygon P whose parity sequence is 'xoeoy' (where x and y are any sequence of 'o's and 'e's), can be transformed into a polygon P' with the parity sequence 'xey', such that $N_{b-w}(P') = N_{b-w}(P)$.

Indeed, suppose that the 3 edges in P corresponding to 'oeo' are e_1, e_2 and e_3. Without loss of generality we may suppose that e_1 is a vertical edge. Suppose e_1, e_2, e_3 and e_4 are adjacent to the five vertices v_1, v_2, v_3, v_4 and v_5. By Lemma 3.4, the turn sequence can be transformed in such a way that v_2, v_3, v_4 corresponds to 'crc' (as in Figure 6(a).).

Define a broken line l the same way as in the proof of Lemma 4. Let $\mathbf{u} = \overrightarrow{v_3v_2}$ and shift every vertex to the left of l by $\mathbf{u}$ as it is shown on Figure 6 (a). There will be no vertex and no edge inside box B'. Suppose we get P_1 by the above transformations. As $\mathbf{u}$ is of even length, clearly $N_{b-w}(P_1) = N_{b-w}(P)$.

Let w be the point where the lines along the edges e_1 and e_4 meet (see Fig. 5(b)). Let e_1^* be the edge v_1w and e_4^* be wv_5. Now replace edges e_1, e_2, e_3 and e_4 by e_1^* and e_4^*. Since the length of e_1^* is the sum of the lengths of e_1 and e_3, both of which were odd, e_1^* is of even length. The length of e_4^* is the sum of the lengths of e_4 and e_2, so since e_2 was of even length, e_4^* has the same parity as e_4.

Figure 7: The polygon depicted in the left-hand (respectively, right-hand) side has 8 (respectively, 12) sides such that if we remove 2 (respectively 3) dashed squares it is not tilable.

Thus, if we get P' from P_1 by changing e_1, e_2, e_3 and e_4 to e_1^* and e_4^*, then the parity sequence changes from 'xoeoy' to 'xey'. The difference in the area of polygon P_1 and P' is box B', whose horizontal side is of even length. Therefore $N_{b-w}(P') = N_{b-w}(P_1) = N_{b-w}(P)$.

We can similarly show that a sequence 'xeey' can be changed into 'xy'. By using the notation on Figure 6(b), suppose now that e_2 and e_3 are of even length, and again that v_2, v_3 and v_4 correspond to 'crc'. In this case we find that e_1^* and e_4^* have the same parity as e_1 and e_4, therefore P' has the parity sequence 'xy'. Again, the difference box B' contains an equal number of black and white squares, therefore $N_{b-w}(P) = N_{b-w}(P')$.

Thus, by applying the above two transformations (changing the parity sequence from 'xoeoy' to 'xey' or from 'xeey' to 'xy') several times, we can either

(a) get rid of all 'e's in the parity sequence, thus getting a polygon P^* which satisfies the conditions of the theorem; or

(b) get a polygon P^* with four sides, at least two of which are of even length. But then $N_{b-w}(P^*) = 0$, thus proving the theorem. ■

Corollary 3.1 *The number $|N_{b-w}(P)|$ may be computed by a linear time procedure using only the parity sequence of P.*

The proof of Corollary 3.1 follows from the proof of Theorem 3.3.

3.3 Additional observations

It is known that if we take a rectangle R of odd sides with $N_{b-w}(R) > 0$, and remove one of its black squares, then the resulting set can always be tiled by dominoes. For the proof one has to observe that the resulting set can always be dissected into 4 or less rectangles each of which has at least two even sides (see [2]).

One can ask if a generalization of this theorem holds: Take a polygon P of $4n$ sides all of whose lengths are odd, and suppose $N_{b-w}(P) > 0$. Remove n black squares from the chessboard coloring. Can the resulting set P' be always tiled by dominoes? The answer for $n \geq 2$ is no, as shown by the counterexample in Figure 7.

The following example also shows that a simple generalization of Theorem 3.1 in 3-dimensional space does not hold. Let a *3d-domino* be a $2 \times 1 \times 1$ polytope. There is a

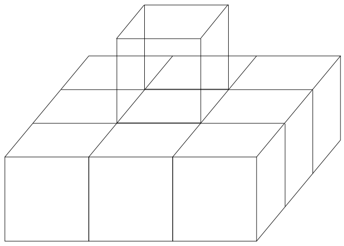

Figure 8: A polytope which can be tiled by $3d$-dominoes but all of whose sides are odd.

polytope in 3 dimensional space all of whose edges are of odd length, but which *can* be tiled by 3d-dominos. The polytope is depicted in Figure 8.

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