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The existence of resolvable Mendelsohn designs RMD($\{3, s^*\}, v$)

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Abstract

In this paper it is proved that, for any positive integer $v \equiv 1$ or $2 \pmod 3$, $v \geq 5$, there exists a resolvable Mendelsohn design where each parallel class consists of blocks of size three and a unique block of size four (when $v \equiv 1 \pmod 3$) or a unique block of size five (when $v \equiv 2 \pmod 3$).

1 Introduction

Let X be a set of v points. A Mendelsohn design of X is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a collection of cyclicsubsets of X (called blocks) such that any ordered pair of distinct points from X occurs together in exactly one block in the collection. In graph-theoretic terms, a Mendelsohn design is equivalent to the decomposition of the complete symmetric directed graph K_v^* on v vertices into circuits. A Mendelsohn design is called resolvable if its block set admits partitions into parallel classes, each parallel class being a partition of the point set.

A Mendelsohn triple system of order v, briefly MTS(v), is a Mendelsohn design $(X, \mathcal{B})$ where $\mathcal{B}$ is a collection of cyclically ordered 3-subsets of X. It is easy to see that the necessary condition for its existence is $v(v-1) \equiv 0 \pmod 3$. An MTS(v) is called resolvable, denoted by RMTS(v), if its block set admits partitions into parallel classes. It is easy to see that the necessary condition for its existence is that v is a multiple of 3.

For RMTS(v), Bermond, Germa and Sotteau have obtained the following result.

Theorem 1.1 [2] *An RMTS(v) exists if and only if $v \equiv 0 \pmod 3$ and $v \neq 6$.*

When v is not a multiple of 3, we can consider resolvable Mendelsohn designs analogously to Ĉerný, Horák and Wallis [5] and Cao and Du [4]. We introduce a resolvable Mendelsohn design which requires each parallel class to consist of blocks of size three and a unique block of size four (when $v \equiv 1 \pmod 3$) or a unique block of size five (when $v \equiv 2 \pmod 3$). We denote these by RMD($\{3,4^*\},v$) or RMD($\{3,5^*\},v$) respectively. Some simple computations show that they contain $v-1$ parallel classes.

In this article we shall investigate the existence of RMD($\{3,4^*\},v$) and RMD($\{3,5^*\},v$). It is proved that these exist for all positive integers $v \equiv 1 \pmod 3$ and $v \geq 7$ for RMD($\{3,4^*\},v$), and all positive integers $v \equiv 2 \pmod 3$ and $v \geq 5$ for RMD($\{3,5^*\},v$).

Theorem 1.2 *An RMD($\{3,4^*\},v$) exists if and only if $v \equiv 1 \pmod 3$ and $v \geq 7$.*

Theorem 1.3 *An RMD($\{3,5^*\},v$) exists if and only if $v \equiv 2 \pmod 3$ and $v \geq 5$.*

2 Preliminaries

In this section we shall define some of the auxiliary designs and establish some of the fundamental results which will be used later. The reader is referred to [3] for more information on designs, and, in particular, Mendelsohn frames and the Oberwolfach problem.

Let X be a set of v points, $\mathcal{G}$ be a partition of X (called holes), and $\mathcal{A}$ be a collection of cyclically ordered 3-subsets of X (called blocks). Suppose there is a set $\mathcal{P}$ of partial parallel classes of X, which satisfies the following properties:

1. Each $P \in \mathcal{P}$ is a partition of $X \backslash G$ for some $G \in \mathcal{G}$, where $P \subseteq \mathcal{A}$.
2. Every ordered pair of points which come from different holes of $\mathcal{G}$ occurs consecutively in exactly one block of some $P \in \mathcal{P}$.
3. $\bigcup_{P \in \mathcal{P}} P = \mathcal{A}$.

Then the triple $(X, \mathcal{G}, \mathcal{A})$ is called a Mendelsohn frame. The type of a Mendelsohn frame is the multiset of size $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description: type $1^i 2^j 3^k \cdots$ denotes i occurrences of holes of size 1, j occurrences of holes of size 2, and so on.

For the Mendelsohn frame, Bennett, Wei and Zhu [1] have obtained the following result.

Theorem 2.1 [1] *A Mendelsohn frame of type g^u exists if and only if $u \geq 4$ and $g(u-1) \equiv 0 \pmod 3$, with possible exceptions for $u = 6$ and $g \in \{3, 21\}$.*

The main technique used here is a variant of Stinson's "Filling in Holes" construction. As the "Filling in Holes" construction will generally involve adjoining more than one infinite point to a frame, the notation for an incomplete resolvable Mendelsohn design is required. Let $v \equiv w \equiv s \pmod 3$, $s = 1$ or 2. An incomplete resolvable Mendelsohn design, IRMD($\{3,(3+s)^*\},v,w$), is a triple $(X,Y,\mathcal{B})$ where X is a set of v points, Y is a subset of X of size w (called a hole) and $\mathcal{B}$ is a collection of cyclically ordered subsets of X (called blocks), each block having size 3 or $3+s$, such that:

1. any ordered pair of distinct points from $X\backslash Y$ occurs together in exactly one block of $\mathcal{B}$;

2. $\mathcal{B}$ admits a partition into $v-w$ parallel classes, each consisting of one block of size $3+s$ and $\frac{v-3-s}{3}$ blocks of size 3 on X, and $w-1$ holey parallel classes, each consisting of $\frac{v-w}{3}$ blocks of size 3 on $X\backslash Y$.

Example 2.2 The following is an IRMD($\{3,5^*\},8,2$):

Point set: $X=Z_8$, $Y=\{6,7\}$.
Parallel classes: $(0,2,1),(3,5,6,4,7);(0,3,6),(1,4,2,5,7);$
$(0,4,3),(1,7,2,6,5);(0,7,5),(1,3,2,4,6);$
$(1,5,4),(0,6,2,3,7);(1,6,3),(0,5,2,7,4).$
Holey parallel classes: (0, 1, 2),(3, 4, 5).

Example 2.3 The following is an IRMD($\{3,4^*\},10,4$):

Point set: $X=Z_{10}$, $Y=\{6,7,8,9\}$.
Parallel classes: $(0,4,6),(1,3,7),(2,8,5,9);(0,5,7),(2,6,3),(1,8,4,9);$
$(0,6,1),(2,4,7),(3,9,5,8);(0,7,4),(1,6,5),(2,9,3,8);$
$(1,7,3),(2,5,6),(0,9,4,8);(2,7,5),(3,6,4),(0,8,1,9).$
Holey parallel classes: $(0,1,2),(3,4,5);(0,2,3),(1,5,4);(0,3,5),(1,4,2).$

The Oberwolfach problem can be applied to the construction of resolvable Mendelsohn designs. A subgraph F of graph G is called a factor of G if F contains all the vertices of G. A 2-factor of G is a factor which is regular of degree 2. A 2-factorization of G is a partition of the edge set of G into 2–factors. More formally, a 2-factorization of G is a pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a collection of edge disjoint 2-factors which partition the edge set of G with the vertex set X. An $(m_1,m_2,\ldots,m_t)$-2-factor of G is a 2-factor consisting of cycles of lengths $m_1,m_2,\ldots,m_t$. An $(m_1,m_2,\ldots,m_t)$-2-factorization of G is a partition of the edge set of G into $(m_1,m_2,\ldots,m_t)$-2-factors.

Suppose n is odd and $n=m_1+m_2+\cdots+m_t$. The problem of determining whether there exists an $(m_1,m_2,\ldots,m_t)$-2-factorization of K_n is the Oberwolfach problem, denoted OP($m_1,m_2,\ldots,m_t$). In this paper our attention is restricted to the case that all cycles are of length three except that each 2-factor contains one cycle of length s, denoted by $OP(\{3,s^*\},n)$. For OP($\{3,4^*\},n$), Dejter, Franek, Mendelsohn and Rosa [6] have obtained the following result.

Theorem 2.4 [6] *There exists an OP($\{3,4^*\},n$) for every $n \equiv 1$ (mod 6) with $n \geq 7$.*

For OP($\{3,5^*\},n$), Sui and the second author of this paper have obtained the following result.

Theorem 2.5 [7] *There exists an OP($\{3,5^*\},n$) for every $n \equiv 5$ (mod 6) with $n \geq 5$ except for $n = 11$.*

3 The existence of RMDs

First we shall give the main construction in this paper. It is a variant of Stinson's "Filling in Holes" construction.

Construction 3.1 Suppose w is a positive integer, $w \equiv 1$ or 2 (mod 3) and $w \geq 4$, and $s = 4$ when $w \equiv 1$ (mod 3) or $s = 5$ when $w \equiv 2$ (mod 3). Suppose

1. there is a Mendelsohn frame of type $g_1 g_2 \cdots g_m$;
2. there is an IRMD($\{3,s^*\}, g_i + w, w$) for every $i < m$;
3. there is an RMD($\{3,s^*\}, g_m + w$).

Then there is an RMD($\{3,s^*\},v$), where $v = \sum_{1 \leq i \leq m} g_i + w$.

Proof We start with a Mendelsohn frame of type $g_1 g_2 \cdots g_m$ $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G} = \{G_1, G_2, \cdots, G_m\}$ and $|G_i| = g_i$ $(1 \leq i \leq m)$. For $i < m$, there are g_i frame holey parallel classes missing the group G_i, and the same number of parallel classes in the IRMD($\{3,s^*\}, g_i + w, w$) which contains a block of size five; match these up arbitrarily, placing the g_i points of the IRMD($\{3,s^*\}, g_i + w, w$) on the i-th group of the frame and the w points in its hole on w new points.

Next, each IRMD($\{3,s^*\}, g_i + w, w$) contains $w - 1$ holey parallel classes. The union of these holey parallel classes together with the $w - 1$ parallel classes of the RMD($\{3,s^*\}, g_m + w$) forms $w - 1$ additional parallel classes. The remaining g_m parallel classes of the RMD($\{3,s^*\}, g_m + w$) can be matched arbitrarily with the g_m frame holey parallel classes of the m-th group. This completes the construction.

It is easy to check that this construction gives the desired designs. The proof is complete.

Next we discuss the two cases: $v \equiv 1$ (mod 3) and $v \equiv 2$ (mod 3). First we consider the existence of RMD($\{3,4^*\},v$) when $v \equiv 1$ (mod 3), $v \geq 7$.

Lemma 3.2 *There exists a RMD($\{3,4^*\},v$) for every $v \equiv 1$ (mod 6) with $v \geq 7$.*

Proof Start with an OP($\{3,4^*\},v$) (for existence, see Theorem 2.4) and for each cycle $\{a,b,c\}$ or $\{x,y,z,w\}$ of the design, we associate the blocks (a,b,c) and (c,b,a) or (x,y,z,w) and (w,z,y,x) of the RMD($\{3,4^*\},v$).

Lemma 3.3 *There exists an RMD*($\{3,4^*\},v$) *for every* $v \in \{10,16,22\}$.

Proof The following is an RMD($\{3,4^*\},10$):

Point set: Z_{10}.

Parallel classes: $(0,1,2),(3,4,5),(6,7,8,9);(0,2,1),(3,5,4),(6,9,8,7);$
$(0,3,6),(1,4,7),(2,8,5,9);(0,4,8),(1,3,9),(2,7,5,6);$
$(0,6,4),(2,5,7),(1,9,3,8);(0,7,9),(2,6,3),(1,5,8,4);$
$(0,9,5),(1,7,3),(2,4,6,8);(1,6,5),(2,9,4),(0,8,3,7);$
$(1,8,6),(4,9,7),(0,5,2,3)$.

For short, an RMD($\{3,4^*\},16$) is constructed in the Appendix. As for an RMD($\{3,4^*\},22$), it is constructed by adding an RMD($\{3,4^*\},7$) to an IRMD($\{3,4^*\},22,7$). An IRMD($\{3,4^*\},22,7$) is constructed in the Appendix and an RMD($\{3,4^*\},7$) is obtained from Lemma 3.2.

Lemma 3.4 *There exists an RMD*($\{3,4^*\},v$) *for every* $v \equiv 4 \pmod 6$ *with* $v \geq 28$.

Proof Start with a Mendelsohn frame of type 6^u with $u \geq 4$ (for existence, see Lemma 2.1), and apply Construction 3.1 with $w = 4$ to obtain the desired designs; the input designs we need, IRMD($\{3,4^*\},10,4$), and RMD($\{3,4^*\},10$) come from Example 2.3 and Lemma 3.3.

Combining Lemma 3.2 to Lemma 3.4, we have the following result.

Theorem 3.5 *There exists an RMD*($\{3,4^*\},v$) *for every* $v \equiv 1 \pmod 3$ *with* $v \geq 7$.

The proof of Theorem 1.2 The necessity obviously holds. The sufficiency comes from Theorem 3.5 and it is easy to see that there exists no RMD($\{3,4^*\},4$).

Next we consider the existence of RMD($\{3,5^*\},v$) when $v \equiv 2 \pmod 3$, $v \geq 5$.

Lemma 3.6 *There exists an RMD*($\{3,5^*\},v$) *for every* $v \equiv 5 \pmod 6$ *with* $v \geq 5$ *except for* $v = 11$.

Proof Start with an OP($\{3,5^*\},v$) (for existence, see Theorem 2.5) and for each cycle $\{a,b,c\}$ or $\{x,y,z,u,v\}$ of the design, we associate the blocks (a,b,c) and (c,b,a) or (x,y,z,u,v) and (v,u,z,y,x) of the RMD($\{3,5^*\},v$).

Lemma 3.7 *There exists an RMD*($\{3,5^*\},v$) *for every* $v \in \{8,11,14,20\}$.

Proof The following is an RMD($\{3,5^*\},8$):
Point set: Z_8.
Parallel classes: $(0,1,2),(3,4,5,6,7);(0,2,1),(3,5,4,7,6);$
$(0,3,6),(1,4,2,5,7);(0,4,3),(1,7,2,6,5);$
$(0,7,5),(1,3,2,4,6);(1,5,3),(0,6,2,7,4);$
$(1,6,4),(0,5,2,3,7)$.

For short, the RMD($\{3,5^*\},11$) and RMD($\{3,5^*\},14$) are constructed in the Appendix. As for RMD($\{3,5^*\},20$), it is constructed by adding an RMD($\{3,5^*\},5$) to an IRMD($\{3,5^*\},20,5$). An IRMD($\{3,5^*\},20,5$) is constructed in the Appendix and an RMD(5) comes from Lemma 3.6.

Lemma 3.8 *There exists an RMD*($\{3,5^*\},v$) *for every* $v\equiv 2 \pmod 6$ *with* $v\geq 26$.

Proof Start with a Mendelsohn frame of type 6^u with $u\geq 4$ (for existence, see Lemma 2.1), and apply Construction 3.1 with $w=2$ to obtain the desired designs; the input designs we need, IRMD($\{3,5^*\},8,2$) and RMD($\{3,5^*\},8$), come from Example 2.2 and Lemma 3.7.

Combining Lemma 3.6 to Lemma 3.8, we have the following result.

Theorem 3.9 *There exists an RMD*($\{3,5^*\},v$) *for every* $v\equiv 2 \pmod 3$ *with* $v\geq 5$.

The proof of Theorem 1.3 The necessity obviously holds. The sufficiency comes from Theorem 3.9.

Appendix

RMD($\{3,5^*\},11$):
Point set: Z_{11}.
Parallel classes:
(0,1,2)(3,4,5)(6,7,8,9,10); (0,2,1)(3,5,4)(6,8,7,10,9);
(0,3,6)(1,4,7)(2,5,9,8,10); (0,4,8)(1,3,9)(2,6,5,10,7);
(0,5,7)(1,6,3)(2,9,4,10,8); (0,6,4)(2,8,5)(1,9,7,3,10);
(0,7,9)(1,10,5)(2,3,8,4,6); (0,9,3)(1,5,8)(2,7,6,10,4);
(1,7,4)(5,6,9)(0,8,3,2,10); (1,8,6)(2,4,9)(0,10,3,7,5).

RMD($\{3,5^*\},14$):
Point set: Z_{14}.
Parallel classes:
(0,1,11)(3,10,12)(4,9,6)(2,7,8,13,5); (0,2,5)(3,7,11)(4,6,9)(1,10,8,12,13);

(0,4,1)(3,5,8)(7,10,9)(2,6,13,12,11); (0,7,13)(1,4,8)(2,3,9)(5,10,11,12,6);
(0,8,9)(1,12,2)(5,6,7)(3,11,10,13,4); (0,9,12)(1,8,5)(6,11,13)(2,10,4,7,3);
(0,10,3)(1,13,9)(4,11,8)(2,12,5,7,6); (0,13,10)(1,7,12)(3,8,6)(2,9,5,11,4);
(1,2,11)(6,12,10)(7,9,8)(0,5,13,3,4); (1,3,6)(2,4,13)(5,9,11)(0,12,8,10,7);
(1,5,3)(4,12,7)(9,13,11)(0,6,10,2,8); (1,9,10)(4,5,12)(6,8,11)(0,3,13,7,2);
(2,13,8)(3,12,9)(4,10,5)(0,11,7,1,6).

RMD($\{3,4^*\},16$):
Point set: Z_{16}.
Parallel classes:
(0,9,10)(1,3,4)(5,15,7)(6,11,14)(2,13,8,12); (1,13,11)(2,10,14)(3,6,7)(5,9,8)(0,15,12,4);
(0,14,8)(1,4,5)(2,6,10)(11,13,15)(3,9,7,12); (0,12,13)(1,2,9)(3,5,8)(7,11,10)(4,14,15,6);
(0,11,3)(1,6,15)(4,8,9)(7,14,13)(2,12,10,5); (0,2,14)(3,7,10)(4,11,8)(5,12,6)(1,15,9,13);
(2,7,4)(5,13,9)(6,14,11)(8,10,15)(0,3,12,1); (0,13,5)(1,7,8)(2,3,11)(9,15,10)(4,6,12,14);
(0,6,9)(1,8,2)(3,10,4)(7,13,14)(5,11,12,15); (0,4,12)(1,5,14)(2,15,3)(7,9,11)(6,13,10,8);
(0,5,7)(1,14,10)(2,8,15)(3,13,6)(4,9,12,11); (0,7,2)(1,9,6)(3,15,14)(8,13,12)(4,10,11,5);
(0,8,11)(2,5,6)(4,7,15)(9,14,12)(1,10,13,3); (0,10,6)(1,12,7)(2,11,9)(4,15,13)(3,8,14,5);
(2,4,13)(3,14,9)(5,10,12)(6,8,7)(0,1,11,15).

IRMD($\{3,5^*\},20,5$):
Point set: $Z_{15}\bigcup\{\infty_i|1\le i\le 5\}$.
Parallel classes: develop the following modulo 15:
(0,2,1,7,3)(4,11,∞_1)(5,13,∞_2)(6,9,∞_3)(12,10,∞_4)(14,8,∞_5).
Holey parallel classes:
(0,1,5)(3,4,8)(6,7,11)(9,10,14)(12,13,2); (1,2,6)(4,5,9)(7,8,12)(10,11,0)(13,14,3);
(2,3,7)(5,6,10)(8,9,13)(11,12,1)(14,0,4); (0,5,10)(1,6,11)(2,7,12)(3,8,13)(4,9,14).

IRMD($\{3,4^*\},22,7$):
Point set: $Z_{15}\bigcup\{\infty_i|1\le i\le 7\}$.
Parallel classes: develop the following modulo 15:
(0,3,1,∞_1)(2,8,∞_2)(4,11,∞_3)(6,5,∞_4)(12,9,∞_5)(13,7,∞_6)(14,10,∞_7).
Holey parallel classes:
(0,1,5)(2,12,13)(3,4,8)(6,7,11)(9,10,14); (0,2,7)(1,9,11)(3,5,10)(4,12,14)(6,8,13);
(0,4,14)(1,11,12)(2,3,7)(5,6,10)(8,9,13); (0,5,13)(1,3,8)(2,10,12)(4,6,11)(7,9,14);
(0,8,10)(1,6,14)(2,4,9)(3,11,13)(5,7,12); (0,10,11)(1,2,6)(3,13,14)(4,5,9)(7,8,12).

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