98. Triangles with integer sides

It is well–known [1–6] that the number T_n of triangles with integer sides and perimeter n is given by

$$T_n = \begin{cases} \langle (n+3)^2/48 \rangle & \text{if } n \text{ is odd} \\ \langle n^2/48 \rangle & \text{if } n \text{ is even} \end{cases}$$

where $\langle x \rangle$ is the integer closest to x.

The object of this note is to give as quick a proof of this as I can.

We prove

Lemma 1.

The number S_n of scalene triangles with integer sides and perimeter n is given for $n \geq 6$ by

$$S_n = T_{n-6}.$$

Proof: If $n = 6,\ 7,\ 8$ or 10, both are 0. Otherwise, given a scalene triangle with integer sides $a < b < c$ and perimeter n, let $a' = a - 1$, $b' = b - 2$, $c' = c - 3$. Then $a',\ b',\ c'$ are the sides of a triangle of perimeter $n - 6$. Moreover, the process is reversible. The result follows.

Corollary.

$$T_n - T_{n-6} = I_n,$$

where I_n denotes the number of isosceles (including equilateral) triangles with integer sides and perimeter n.

Lemma 2. If $n \geq 1$

$$I_n = \begin{cases} (n-4)/4 & \text{if } n \equiv 0 \pmod 4 \\ (n-1)/4 & \text{if } n \equiv 1 \pmod 4 \\ (n-2)/4 & \text{if } n \equiv 2 \pmod 4 \\ (n+1)/4 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof: If $n = 1,\ 2$ or 4, $I_n = 0$. Otherwise, if $n \equiv 0 \pmod 4$, write $n = 4m$. The isosceles triangles with integer sides and perimeter n have sides

$$\{2,\ 2m-1,\ 2m-1\},\ \{4,\ 2m-2,\ 2m-2\},\ \cdots\ ,\{2m-2,\ m+1,\ m+1\}.$$

Thus there are $m - 1$ such triangles, and $I_n = m - 1 = (n-4)/4$. The other three cases are similar.

Lemma 3. For $n \geq 7$

$$I_n + I_{n-6} = \begin{cases} (n-6)/2 & \text{if } n \text{ is even} \\ (n-3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Suppose $n \equiv 0 \pmod 4$. Then $n-6 \equiv 2 \pmod 4$, $I_n = (n-4)/4$, $I_{n-6} = (n-8)/4$ and $I_n + I_{n-6} = (n-6)/2$.
If $n \equiv 2 \pmod 4$, $n-6 \equiv 0 \pmod 4$, $I_n + I_{n-6} = (n-2)/4 + (n-10)/4 = (n-6)/2$.
So if n is even, $I_n + I_{n-6} = (n-6)/2$.
The case n odd is similar.

Lemma 4. For $n \geq 12$

$$T_n - T_{n-12} = \begin{cases} (n-6)/2 & n \text{ even} \\ (n-3)/2 & n \text{ odd.} \end{cases}$$

Proof:

$$T_n - T_{n-6} = I_n, \quad T_{n-6} - T_{n-12} = I_{n-6}, \quad T_n - T_{n-12} = I_n + I_{n-6}.$$

Lemma 5. Let $f(n)$ be defined by

$$f(n) = \begin{cases} n^2/48 & n \text{ even} \\ (n+3)^2/48 & n \text{ odd.} \end{cases}$$

Then

$$f(n) - f(n-12) = \begin{cases} (n-6)/2 & n \text{ even} \\ (n-3)/2 & n \text{ odd.} \end{cases}$$

Lemma 6. Let $\delta_n = T_n - f(n)$. Then for $n \geq 12$

$$\delta_n = \delta_{n-12}.$$

Theorem.

$$T_n = \langle f(n) \rangle.$$

Proof: It is easy to check that $|\delta_n| \leq 1/3$ for $0 \leq n \leq 11$, so by Lemma 6, $|\delta_n| \leq 1/3$ for all n. The result follows.

Reference

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[3] R. Honsberger, *Mathematical Gems* III, vol. 9, Dolciana Mathematical Expositions, Mathematical Association of America, Washington, DC, 1985.

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For the referee

n	0	1	2	3	4	5	6	7	8	9	10	11
T_n	0	0	0	1	0	1	1	2	1	3	2	4
T_{n-6}							0	0	0	1	0	1
I_n	0	0	0	1	0	1	1	2	1	2	2	3
$f(n)$	0	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{25}{12}$	$\frac{4}{3}$	3	$\frac{25}{12}$	$\frac{49}{12}$
δ_n	0	$-\frac{1}{3}$	$-\frac{1}{12}$	$\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{12}$	$-\frac{1}{3}$	0	$-\frac{1}{12}$	$-\frac{1}{12}$