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TRIANGLES WITH INTEGER SIDES, REVISITED

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Introduction Our attention has once more been drawn (see [**2**] and its references) to the problem of determining the number T_n of triangles with integer sides and perimeter n. The solution of this problem can be written neatly as

$$T_n = \begin{cases} \left\langle \dfrac{(n+3)^2}{48} \right\rangle & \text{for } n \text{ odd;} \\ \left\langle \dfrac{n^2}{48} \right\rangle & \text{for } n \text{ even,} \end{cases}$$

where $\langle x \rangle$ is the integer closest to x. Our proof comes in two stages. First we show by a direct combinatorial argument that

$$T_n = \begin{cases} p_3\left(\dfrac{n-3}{2}\right) & \text{for } n \text{ odd;} \\ p_3\left(\dfrac{n-6}{2}\right) & \text{for } n \text{ even,} \end{cases}$$

where $p_3(n)$ is the number of partitions of n into at most three parts. Then we show using a novel partial fractions technique that

$$p_3(n) = \left\langle \frac{(n+3)^2}{12} \right\rangle .$$

The proofs For a triangle with integer sides $a \le b \le c$ and odd perimeter n,

$$a+b-c, \quad b+c-a, \quad c+a-b$$

are odd and positive,

$$a+b-c-1, \quad b+c-a-1, \quad c+a-b-1$$

are even and nonnegative, and if

$$p = \frac{1}{2}(b+c-a-1), \quad q = \frac{1}{2}(c+a-b-1), \quad \text{and} \quad r = \frac{1}{2}(a+b-c-1),$$

then $p \geq q \geq r$ are nonnegative integers, and $p+q+r = \dfrac{n-3}{2}$.

Conversely, if $n \geq 3$ is odd and $p \geq q \geq r$ are nonnegative integers with $p+q+r = \dfrac{n-3}{2}$ and if

$$a = q+r+1, \quad b = p+r+1, \quad \text{and} \quad c = p+q+1,$$

then $a \leq b \leq c$ are the sides of a triangle with perimeter n.

Similarly, given a triangle with integer sides $a \leq b \leq c$ and even perimeter n,

$$a+b-c, \quad b+c-a, \quad \text{and} \quad c+a-b$$

are even and positive,

$$a+b-c-2, \quad b+c-a-2, \quad \text{and} \quad c+a-b-2$$

are even and nonnegative, and if

$$p = \frac{1}{2}(b+c-a-2), \quad q = \frac{1}{2}(c+a-b-2), \quad \text{and} \quad r = \frac{1}{2}(a+b-c-2),$$

then $p \geq q \geq r$ are nonnegative integers, and $p+q+r = \dfrac{n-6}{2}$.

Conversely, if $n \geq 6$ is even and $p \geq q \geq r$ are nonnegative integers with $p+q+r = \dfrac{n-6}{2}$ and if

$$a = q+r+2, \quad b = p+r+2, \quad \text{and} \quad c = p+q+2$$

then $a \leq b \leq c$ are the sides of a triangle with perimeter n. ■

To show that

$$p_3(n) = \left\langle \frac{(n+3)^2}{12} \right\rangle,$$

we start with the generating function

$$\sum_{n \geq 0} p_3(n) q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

To see that this is indeed the generating function for partitions into at most three parts, we note that partitions into at most three parts are equinumerous with